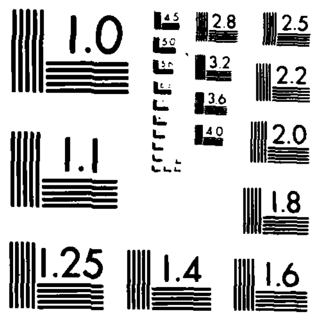
AD-A142 793 NUMERICAL SOLUTION OF VISCOPLASTIC FLOW PROBLEMS BY AUGMENTED LAGRANGIANS(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER P LE TALLEC MAY 84 1/1

UNCLASSIFIED MRC-TSR-2690 DAAG29-80-C-0041 F/G 12/1 NL

END
DATE FILMED
8-84
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

AD-A142 793

MRC Technical Summary Report #2690

NUMERICAL SOLUTION OF VISCOPLASTIC FLOW PROBLEMS BY AUGMENTED LAGRANGIANS

Patrick Le Tallec

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

May 1984

(Received January 12, 1984)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

NUMERICAL SOLUTION OF VISCOPLASTIC FLOW PROBLEMS BY AUGMENTED LAGRANGIANS

Patrick Le Tallec

Technical Summary Report #2690
May 1984

ABSTRACT

This report describes an application of Augmented Lagrangian techniques to the numerical solution of quasistatic flow problems in incompressible viscoplasticity, focusing on cases where the internal viscoplastic dissipation potential is not a differentiable function of the material deformation rate. The stresses of elastic origin are neglected, and the variational formulation of these problems is approximated via mixed finite elements of order 1. Convergence results are proved or recalled, both for the finite element approximation and for the augmented lagrangian algorithm. A detailed study of the local minimization problems which occur in the augmented lagrangian decomposition of the above problems is also presented, together with several numerical results. These results were obtained using the MODULEF finite element code on a VAX 780 at the Mathematics Research Center and cover successively the case of Norton, of Bingham and of Tresca type materials.

AMS (MOS) Subject Classifications: 65K10, 65N30, 73F05, 76A05

Key Words: viscoplasticity, convexity, incompressibility, finite elements, augmented lagrangians.

Work Unit Number 3 (Numerical Analysis and Scientific Computing)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and in part by the Laboratoire Central des Ponts et Chaussees, 58 boulevard LEFEBVRE, 75015 PARIS, FRANCE.

SIGNIFICANCE AND EXPLANATION

Augmented lagrangian methods, introduced around 1970 by M. R. Hestenes and M. J. D. Powell, are now classical numerical tools in scientific computation. They take into account the dual structure that most problems in continuum mechanics do present, involving usually both stresses and displacements (or velocities), to reformulate them as saddle-point problems, which can then be solved numerically by Uzawa type algorithms. These methods have already been used in situations like viscoplasticity by GLOWINSKI and MAROCCO [1975] and are described in detail in FORTIN and GLOWINSKI [1982]. Compared to previous publications, this report:

(i) tries to present a clean and updated version of these techniques,

(ii) uses a low order, convergent finite element for the approximation of incompressible velocity fields,

(iii) and studies in details each local minimization problem which appears during the algorithm.

The main mathematical tool used herein will be convex analysis. The goal of this report is to give a comprehensive presentation of all the theoretical aspects which are behind the application of augmented lagrangian techniques to viscoplasticity (existence theory, approximation, convergence of the algorithm, ...) so that the reader may be able to implement these techniques in any finite element code, to obtain reasonable numerical results with a minimal experimentation time, to assess the validity of his numerical results and to judge the efficiency of his numerical technique.

TABLE OF CONTENTS

NUMERICAL SOLUTION OF VISCOPLASTIC FLOW PROBLEMS BY AUGMENTED LAGRANGIANS

Patrick Le Tallec

1. INTRODUCTION AND FORMULATION OF THE CONTINUOUS PROBLEMS.

1. Introduction. We consider in this report the problem of computing the quasistatic flows of incompressible viscoplastic materials subjected to given distributions of external loads. The constitutive law which modelizes the behavior of the considered viscoplastic materials and the configuration of the body are supposed to be given. The unknown is the velocity field inside the body resulting from the application of the external loads.

The materials which are involved in such problems include freshly mixed concrete, bitumen, frozen soils, different types of mud, polymers at high temperature or very hot metals. These materials, when subjected to external loads, flow viscously in a nonreversible pattern and develop stresses which are mainly of viscous origin. Most of these materials flow in an incompressible or nearly incompressible way.

Herein, to compute the velocity field v, we use a variational formulation of the mechanical problem (Sec. 1), which neglects the stresses of elastic origin, we discretize the space of kinematically admissible incompressible velocity fields by mixed finite elements of order 1 (Sec. 2), and finally we solve the resulting discrete problem by augmented lagrangian techniques (Sec. 3). Convergence results are proved both for the finite element approximation and for the augmented lagrangian algorithm, and the local problems which appear in the augmented lagrangian decomposition are studied in details in Sec. 4. Several numerical results are presented in Secs. 5 to 7, successively for Norton, Bingham and Tresca type materials. The basic assumption in this work is that the internal dissipation potential associated to the considered viscoplastic material is a convex,

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and in part by the Laboratoire Central des Ponts et Chaussees, 58 boulevard LEFEBVRE, 75015 PARIS, FRANCE.

continuous but not necessarily differentiable function of the deformation rate tensor inside the body.

1.2 The mechanical problem. Depending whether we consider a specific piece of material with very little motion or a specific domain with incoming and outcoming material, the configuration Ω given in the data of the problem will correspond either to the reference configuration or to the present configuration of the body. In this report, we will suppose that it corresponds to the *reference configuration* of the body; in other words, we will consider solids in small strains. The other case, associated to viscoplastic fluids flowing viscously, is identical within the replacement of the lagrangian coordinates x by the eulerian coordinates X.

Within this convention, the unknown velocity field is determined by the two *mechanical equations below* (PERZYNA [1966]):

constitutive law (viscoplastic incompressible solid in small strains)

$$\begin{cases} (\sigma(x) + p\ \mathrm{Id}) \in \partial\, \mathcal{D}_1(x, \dot{E}(v)), \\ \\ \mathrm{Tr}\ (\dot{E}(v)) = 0,\ \dot{E}(v) = \tfrac{1}{2}(\nabla v + \nabla v^T), \end{cases}$$

virtual work theorem (quasistatic case)

$$\begin{cases} \int_\Omega \sigma\cdot(\nabla w + \nabla w^T)/2\ dx = \int_\Omega f\cdot w\ dx + \int_{\Gamma_2} g\cdot w\ da, \\ \\ \text{for any } w \text{ such that } w = 0 \text{ on } \Gamma_1. \end{cases}$$

These equations involve the Cauchy stress tensor field $\sigma(x)$ and a hydrostatic pressure field $p(x)$. Here, the notations ∇w and $\partial\, \mathcal{D}_1$ represent the gradient of the vector field w and the subgradient of the convex function $\mathcal{D}_1(x,\cdot)$, respectively. In addition, Γ_1 and Γ_2 denote the parts of the boundary of Ω where imposed velocities

$\dot{u}_o$ and imposed tractions g are applied, respectively. Moreover, for x fixed, the internal dissipation potential $\mathcal{D}_1(\cdot)$ is a known convex function of the time derivative of the linearized strain tensor $\mathbf{E}$. This function, defined here over the space of symmetric tensors of $\mathbf{R}^{N\times N}$ with zero trace only depends on the properties of the considered viscoplastic material at point x. For example, if we omit the argument x for simplicity, Norton and Bingham materials are characterized respectively by

(1.1) $$\mathcal{D}_1(\mathbf{D}) = \frac{1}{p}(k\sqrt{2})^p\ |\mathbf{D}|^p \quad \text{(Norton)}$$

(1.2) $$\mathcal{D}_1(\mathbf{D}) = \mu\ |\mathbf{D}|^2 + \sqrt{2}\ g\ |\mathbf{D}|. \quad \text{(Bingham)}.$$

1.3 Variational formulation. If we restrict the virtual work theorem to divergence-free test functions w and if we eliminate the Cauchy stress tensor σ using the constitutive law, then the mechanical equations above correspond, at least formally, to the variational problem:

(1.3) Minimize the dissipated energy rate $J(w)$ over the set K of incompressible kinematically admissible velocity fields,

where J and K are respectively defined by

(1.4) $$\boxed{J(w) = \int_\Omega \mathcal{D}_1(\tfrac{1}{2}(\nabla w+\nabla w^T))\ dx - \int_\Omega f\cdot w\ dx - \int_{\Gamma_2} g\cdot w\ da,}$$

(1.5) $$\boxed{K = \{w \in W^{1,p}(\Omega),\ \text{div}\ w = 0,\ w = \dot{u}_o \ \text{on}\ \Gamma_1\}\quad .}$$

This variational problem is well-posed and we have:

EXISTENCE THEOREM: Let Ω be open bounded connected in $\mathbf{R}^N$ (N=2 or 3) with Lipschitz continuous boundary Γ. We suppose that the interior of Γ_1 is not empty, that $\dot{u}_o$ is the trace of a function of $W^{1,p}(\Omega)$, and satisfies

$$\int_\Gamma \dot{\mathbf{u}}_o \cdot \nu \, da = 0$$

whenever $\Gamma_1 = \Gamma$. We assume moreover that the external body forces $\mathbf{f}$ and surface tractions g are respectively in $L^{p^*}(\Omega)$ and $L^{p^*}(\Gamma_2)$ $(pp^* = p + p^*)$, and that the convex internal dissipation potential $\mathcal{D}_1$ satisfies:

$$(1.6) \qquad c_1 \, |D|^p \le \mathcal{D}_1(D) \le c_2 + c_3 \, |D|^p,$$

almost everywhere in Ω for any symmetric, $N \times N$ matrix D with zero trace, $1 < p < +\infty$.

Then, there exists a velocity field $\mathbf{v}$ which minimizes the dissipated energy rate $J(\mathbf{w})$ over the set K of kinematically admissible velocity fields. This solution is unique if $\mathcal{D}_1$ is strictly convex. Moreover, for each minimizer $\mathbf{v}$, there exists a deviatoric stress tensor field σ_D in $(L^{p^*}(\Omega))^{N\times N}$, a hydrostatic pressure field p in $L^{p^*}(\Omega)$ which satisfy the weak equilibrium equations and constitutive laws:

$$(1.7) \quad \begin{cases} \int_\Omega (\sigma_D - p\mathrm{Id})\cdot D(\mathbf{w}) \, dx = \int_\Omega \mathbf{f}\cdot\mathbf{w} \, dx + \int_{\Gamma_2} g\cdot\mathbf{w} \, da \ , \ \forall\, \mathbf{w} \in V, \\ \sigma_D \in \partial\mathcal{D}_1 \ (\dot{\mathbf{E}}(\mathbf{v})) \ \text{a.e. in } \Omega, \\ V = \{\mathbf{w} \in W^{1,p}(\Omega), \ \mathbf{w} = 0 \ \text{ on } \Gamma_1 \}. \end{cases}$$

Proof: The proof of this result is very classical in convex analysis. The existence of a solution $\mathbf{v}$ involves the Weierstrass theorem, its characterization by (1.7) uses duality arguments and the closed range theorem.

First, from (1.6) and from the Korn's inequality on V (GEYMONAT, SUQUET[1983]), J satisfies

$$c_1 \|\mathbf{w}\|^p_{1,p} - (\|f\|_{p^*} + \|g\|_{p^*})\|\mathbf{w}\|_{1,p} \le J(\mathbf{w}) \le c_2(\mathrm{mes}(\Omega)) + c_3 \|\mathbf{w}\|^p_{1,p} + (\|f\|_{p^*} + \|g\|_{p^*})\|\mathbf{w}\|_{1,p}$$

for any $\mathbf{w}$ in K. Therefore, J is coercive, convex (strictly convex if $\mathcal{D}_1(\cdot)$ is) and continuous on K for the $\mathbf{W}^{1,p}(\Omega)$ topology. It is thus weakly lower semicontinuous on K. In addition, K, defined as the Kernel of the linear application $\mathbf{w} \to \{\mathrm{div}\ \mathbf{w},\ \mathbf{w} - \dot{\mathbf{u}}_o|_{\Gamma_1}\}$, is convex and closed in $\mathbf{W}^{1,p}(\Omega)$. Since K is also not empty, applying the Weierstrass theorem, there exists a minimizer $\mathbf{v}$ of $J(\cdot)$ over K, which is unique if J is strictly convex.

To further characterize such a minimizer $\mathbf{v}$, we now introduce

$$X = \{\mathbf{w} \in \mathbf{W}^{1,p}(\Omega),\ \mathbf{w} = 0 \text{ on } \Gamma_1,\ \mathrm{div}\ \mathbf{w} = 0\},$$

$$Y = \{\mathbf{D} \in (L^p(\Omega))^{N\times N},\ \mathbf{D}^T = \mathbf{D},\ \mathrm{Tr}\ (\mathbf{D}) = 0 \text{ a.e. in } \Omega\},$$

$$\Phi(\mathbf{w},\mathbf{D}) = \int_\Omega \mathcal{D}_1(\mathbf{D}(\mathbf{v}+\mathbf{w}) - \mathbf{D})\ dx - \int_\Omega \mathbf{f}\cdot(\mathbf{v}+\mathbf{w})\ dx - \int_{\Gamma_2} \mathbf{g}\cdot(\mathbf{v}+\mathbf{w})\ da\ .$$

Above $\mathbf{D}(\mathbf{v}+\mathbf{w})$ represents the tensor $\mathbf{D}(\mathbf{v}+\mathbf{w}) = \frac{1}{2}(\nabla(\mathbf{v}+\mathbf{w}) + \nabla(\mathbf{v}+\mathbf{w})^T)$ and Y is in duality with the space

$$Y^* = \{\tau \in (L^{p^*}(\Omega))^{N\times N},\ \tau^T = \tau,\ \mathrm{Tr}\ (\tau) = 0\},$$

through the duality pairing

$$< \tau, \mathbf{D} > = \int_\Omega \tau\cdot\mathbf{D}\ dx = \int_\Omega \sum_{i,j=1}^N \tau_{ij}\ D_{ij}\ dx\ .$$

Obviously, from (1.6), $\Phi(\cdot,\cdot)$ takes on finite values and is real, convex and continuous on XxY. Moreover, since $\mathbf{v}$ *minimizes* J *over* K, 0 is a solution of the primal problem: <u>Minimize</u> $\Phi(\mathbf{w},0)$ <u>on</u> X. From a basic theorem of convex analysis (EKELAND-TEMAM [1976, p 52-53]), this implies that the dual *problem:* <u>Maximize</u> $-\Phi^*(0,\tau)$ <u>over</u> Y^* has a solution $(-\sigma_D)$ which satisfies

(1.8) $\{0, -\sigma_D\} \in \partial\Phi(0,0)$

that is $< -\sigma_D, \mathbf{D} > \leq \Phi(\mathbf{w},\mathbf{D}) - \Phi(0,0),\ \forall\ \{\mathbf{w},\mathbf{D}\} \in X\times Y.$

Writing (1.8) *successively for* $\{\mathbf{w},\mathbf{D}\} = \{\mathbf{w},\mathbf{D}(\mathbf{w})\}$ and $\{\mathbf{w},\mathbf{D}\} = \{0,-\mathbf{B}\}$, we obtain

(1.9) $L(w) = \int_\Omega \sigma_D \cdot D(w)\, dx - \int_\Omega f \cdot w\, dx - \int_{\Gamma_2} g \cdot w\, da = 0,\ \forall\, w \in X,$

(1.10) $\int_\Omega \sigma_D \cdot H\, dx < \int_\Omega \{\mathcal{D}_1(D(v) + H) - \mathcal{D}_1(D(v))\}\, dx,\ \forall\, H \in Y.$

But (1.10) can only hold if (EKELAND-TEMAM [1976 p 21, p 271])

(1.11) $\sigma_D \in \partial \mathcal{D}_1(D(v))$, a.e. in Ω.

Now, to obtain (1.7) out of (1.9), (1.11), it is sufficient to observe that the divergence operator is a continuous surjection from V onto $L^p(\Omega)$ (or onto $L^p(\Omega)/\mathbb{R}$ if $\Gamma_1 = \Gamma$). Therefore, from the closed range theorem, its transpose is a continuous homemorphism from $L^{p^*}(\Omega)$ onto the orthogonal of its Kernel in V^*, that is onto X^*. Since, from (1.9) $L(\cdot)$ is an element of X^*, there exists then an element p in $L^{p^*}(\Omega)$ such that

$L(w) = \langle p, \operatorname{div} w \rangle,\ \forall\, w$ in V,

and our proof is complete. $\square$

<u>REMARK 1.1</u>: We are not supposing here any differentiability of the internal dissipation potential $\mathcal{D}_1(\cdot)$. The numerical techniques to be used later will have to be able to handle such a lack of differentiability. $\square$

<u>REMARK 1.2</u>: Even though the argument x has been omitted in the potential $\mathcal{D}_1$ for simplicity, the whole theory developed in this report applies for potentials which are measurable functions of x on Ω. $\square$

<u>REMARK 1.3</u>: For Norton and for Bingham materials, the internal dissipation potential is strictly convex and satisfies (1.7) with

$$C_2 = 0, \quad C_1 = C_3 = \frac{1}{p}(k\sqrt{2})^p, \quad \text{(Norton)}$$

$$C_1 = \mu, \quad C_2 = \sqrt{2}\, g,\ C_3 = (\sqrt{2}\, g + \mu),\ p = 2 \quad \text{(Bingham)}.$$

But (1.7) is still valid, and therefore the above existence theorem still applies for materials associated to non-strictly convex and non-differentiable potentials such as

$$\mathcal{D}_1 (\mathbf{D}) = \frac{1}{p}(k\sqrt{2})^p \sup_{i,j} (|D_i - D_j|)^p,$$

where D_i are the eigenvalues of the deformation rate tensor $\mathbf{D}$. This corresponds to Tresca's type viscoplasticity.

2. THE DISCRETE PROBLEMS.

2.1 The discrete spaces. The approximation of the set K of kinematically admissible velocity fields, which is needed for the numerical solution of the variational problem (1.4), can not be achieved by the basic finite element spaces used in general. For example, the space of divergence-free functions whose restriction to each triangle (tetrahedron if $N = 3$) of a given regular triangulation of Ω is a first degree polynomial may only approximate a small part of the space of divergence-free elements of $\mathbf{W}^{1,p}(\Omega)$. Therefore, it is a very inappropriate finite dimensional approximation of the set K of kinematically admissible incompressible velocity fields. To obtain a satisfactory approximation of K, the set of approximate test functions must be enriched and the incompressibility constraint must be weakened.

As pointed out in BREZZI [1974] and summarized in GIRAULT-RAVIART [1979] in their study of the Stokes problem, a good approximation of K is obtained as follows:

(i) we first decompose the domain Ω into a regular triangulation $\mathcal{T}_h$ of N_h polygons (N=2) or polyhedrons (N=3) which satisfy the classical assembly conditions described in CIARLET [1978 p 51];

(ii) we then define the space V_h of approximate test functions by

$$(2.1) \quad V_h = \{\mathbf{w}_h \in C^0(\bar{\Omega}),\ \mathbf{w}_h = 0 \text{ on } \Gamma_1,\ \mathbf{w}_h|_{\Omega_\ell} \in \mathbf{P}_x(\Omega_\ell),\ \forall\, \ell = 1, N_h\},$$

where $P_x(\Omega_\ell)$ is a given finite dimensional space of continuous interpolating functions defined on Ω_ℓ;

(iii) in addition, we introduce an appropriate finite element space P_h, included in $L^\infty(\Omega)$ and which satisfies the so-called BREZZI (or inf-sup) condition:

$$(2.2)\quad \boxed{\inf_{\substack{q_h \in P_h \\ q_h \neq 0}} \ \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{\int_\Omega q_h \operatorname{div} w_h \, dx}{|q_h|_{o,p^*} |w_h|_{1,p}} > \beta > 0,}$$

where β is independent of the diameter h of the triangulation T_h, and where p is the exponent which appears in the definition of $K(pp^* = p + p^*)$;

(iv) we finally approximate K by:

$$(2.3)\quad \boxed{K_h = \{w_h,\ (w_h - \dot{u}_o) \in V_h,\ \int_\Omega q_h \operatorname{div} w_h \, dx = 0,\ \forall q_h \in P_h\}\ .}$$

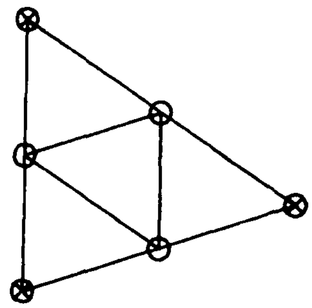
Briefly speaking, this construction of K_h amounts to impose the incompressibility constraint in an <u>averaging sense</u> only. In that way, more elements of V_h can satisfy this constraint and the set K_h is bigger. It can then better approximate K.

The choice of the polyhedrons Ω_ℓ, of the interpolating space $P_x(\Omega_\ell)$ and of the space P_h of approximate pressures is free, provided that the BREZZI condition (2.2) is satisfied. In this report, we will use triangles (respectively tetrahedrons if $N = 3$) as polygons Ω_ℓ, and define $P_x(\Omega_\ell)$ and P_h by

$$(2.4)\quad P_x(\Omega_\ell) = \{w \in C^o(\Omega_\ell),\ w_{|\Omega_\ell^i} \in P_1(\Omega_\ell^i),\ \forall i = 1, 2^N\},$$

$$(2.5)\quad P_h = \{q \in C^o(\bar{\Omega}),\ q_{|\Omega_\ell} \in P_1(\Omega_\ell),\ \forall \ell = 1, N_h\},$$

where $P_1(\Omega_k)$ is the space of first order polynomials defined over Ω_k and where (Ω_ℓ^i) are the 2^N triangles



x position of the degrees of freedom for the pressures

o position of the degrees of freedom for velocities

Figure 2.1 Decomposition of a triangle Ω_ℓ in four equal subtriangles

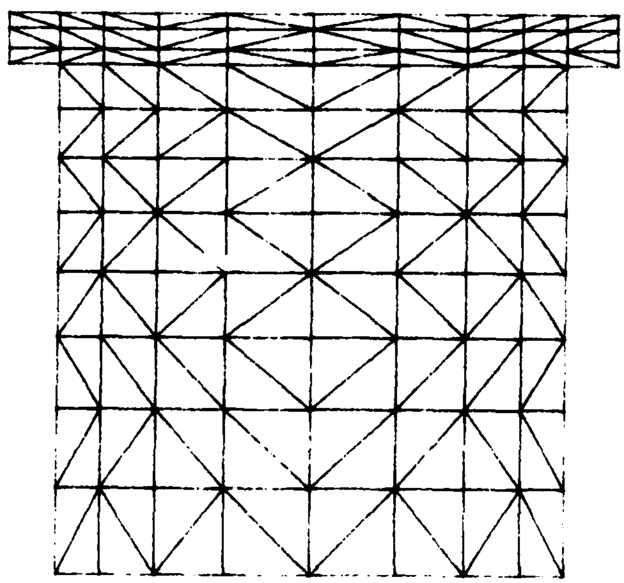


Figure 2.2 Triangulation T_h (pressures)

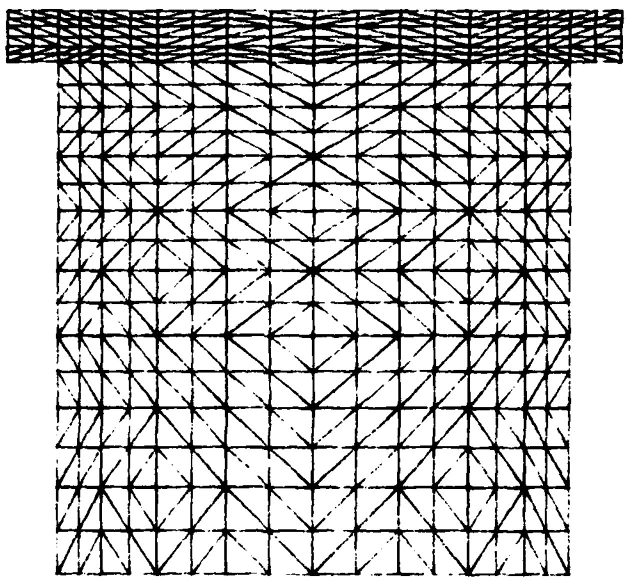


Figure 2.3 Triangulation T_h^2

(velocities)

(respectively tetrahedrons) included in Ω_ℓ which are obtained by joining together the midsides of every edge of Ω_ℓ. With that definition of $P_x(\Omega_\ell)$, the space V_h of approximate test functions is now simply the space of continuous vector functions with zero trace on Γ_1 and whose restriction to each triangle (respectively tetrahedron) of T_h^2 is a first degree polynomial, T_h^2 being the triangulation obtained by dividing each triangle (respectively tetrehedron) of T_h into four equal subtriangles (respectively eight subtetrahedrons). As for the space P_h of approximate pressures, it becomes the space of continuous scalar functions whose restriction to each triangle (respecively tetrahedron) of T_h is a first degree polynomial (see GLOWINSKI [1984] for more details on those discrete spaces).

The above choice of approximate spaces ((2.1), (2.3), (2.4), (2.5)) is far from being the only possible one but it satisfies the BREZZI condition (2.2) and leads to a very convenient approximate augmented lagrangian decomposition of our viscous flow problem (1.3). Moreover, it uses low order finite elements, which is adviseable in nonlinear problems where little regularity is to be expected. Finally, the sets (K_h) constructed by (2.1), (2.3), (2.4) and (2.5) form a converging sequence of finite dimensional approximation of K and we have (BERCOVIER-PIRONNEAU [1977]):

$$(2.6) \qquad \forall\, w \in K,\ \lim_{h \to 0} \left\{ \operatorname*{Inf}_{w_h \in K_h} \| w - w_h \|_{1,p} \right\} = 0 ,$$

REMARK 2.1: When the maximal diameter h of the triangulation goes to zero, we also have (CIARLET [1978]):

$$(2.7) \quad \begin{cases} V_h \subset V \cap W^{1,q}(\Omega),\ \forall\, 1 \le q \le +\infty;\ \ \dim\ V_h < +\infty; \\ \\ \forall w \in V,\ \lim_{h \to 0} \left\{ \operatorname*{Inf}_{w_h \in V_h} \| w - w_h \|_{1,p} \right\} = 0; \end{cases}$$

$$
(2.8)\quad \begin{cases} P_h \subset L^q(\Omega),\ \forall\ 1 \leq q < +\infty;\ \dim\ P_h < +\infty; \\ \\ \forall q \in L^{p^*}(\Omega),\ \lim\limits_{h\to 0}\ \{\inf\limits_{q_h \in P_h} \|q - q_h\|_{o,p^*}\} = 0. \end{cases}
$$

But, since we are imposing the incompressibility constraint in an averaging sense only, K_h is not included in K. □

2.2 The discrete problems. The approximate incompressible viscous flow problem is simply obtained by replaing K by K_h in (1.3). But, since K_h is not included in K, we first have to extend the internal dissipation potential $\mathcal{D}_1(.)$, initially defined as a convex continuous coercive function on the space of symmetric $N \times N$ real matrices with zero trace, to a convex continuous coercive function $\mathcal{D}_1^e(.)$ defined on the whole space of symmetric $N \times N$ real matrices. This extension must be convex and satisfy

$$
(2.9)\quad \begin{cases} \mathcal{D}_1^e\ (D) = \mathcal{D}_1(D),\ \forall\ D \in R_s^{N\times N} \text{ with } \operatorname{Tr} D = 0; \\ \mathcal{D}_1^e(D+qId) \geq \mathcal{D}_1(D),\ \forall q \in R,\ \forall\ D \in R_s^{N\times N} \text{ with } \operatorname{Tr} D = 0; \\ c_1|D|^p \leq \mathcal{D}_1^e\ (D) \leq c_2 + c_3\ |D|^p,\ \forall\ D \in R_s^{N\times N},\ \text{a.e. in } \Omega\ . \end{cases}
$$

In other words, the extension $\mathcal{D}_1^e(\cdot)$ of $\mathcal{D}_1(\cdot)$ coincides with $\mathcal{D}_1(\cdot)$ on the space of symmetric matrices with zero trace, penalizes the slightly compressible velocity fields and extends to $R_s^{N\times N}$ the coercivity and the continuity of the function $\mathcal{D}_1(\cdot)$. The introduction of $\mathcal{D}_1^e(\cdot)$ does not affect the solutions of the viscous flow problems (1.3) since $\mathcal{D}_1^e(\cdot)$ and $\mathcal{D}_1(\cdot)$ coincide for symmetric matrices with zero trace; it only provides a mathematical tool for calculating the dissipation potential for compressible velocity fields and therefore enables us to compute reasonable nearly incompressible finite dimensional approximations of the solutions v of (1.3). The introduction of such extensions $\mathcal{D}_1^e(\cdot)$ which satisfy (2.9) is easy. For example, for Norton and Bingham

materials, the expressions of $\mathcal{D}_1(\cdot)$ given in (1.1) and (1.2) define such extensions. In other cases, one can take

$$\mathcal{D}_1^e(\mathbf{D}) = \mathcal{D}_1(\mathbf{D} - \mathrm{Tr}(\mathbf{D})\mathrm{Id}) + C_1 \, |\mathrm{Tr}\,(\mathbf{D})|^p,$$

being careful not to choose too big values for C_1, in order to avoid "locking" phenomena.

Once this extension defined, the discrete variational formulation of our incompressible viscous flow problems (1.3) is

(2.10) $\boxed{\text{Minimize the dissipated energy rate } J(\mathbf{w}_h) \text{ over the set } K_h \text{ of approximate kinematically admissible velocity fields,}}$

where K_h is the subset which is defined by (2.1), (2.3), (2.4), (2.5) and where the dissipated energy rate $J(\cdot)$, given by (1.4), is extended to a function from K_h into $\mathbb{R}$ by replacing the internal dissipation potential $\mathcal{D}_1(\cdot)$ by its extension $\mathcal{D}_1^e(\cdot)$ introduced in (2.9).

THEOREM 2.1: Under the assumptions of Theorem 1.1, for any fixed h, the discrete incompressible viscous flow problem (2.10) has a solution $\mathbf{v}_h$. Moreover, any solution $\mathbf{v}_h$ of (2.10) is associated to an approximate stress tensor field σ_h and to an approximate pressure field p_h in P_h, such that

$$(2.11)\quad \begin{cases} (\sigma_h + p_h\,\mathrm{Id}) \in \partial\,\mathcal{D}_1^e(\dot{\mathbf{E}}_h) \text{ a.e. on } \Omega, \quad \dot{\mathbf{E}}_h = \tfrac{1}{2}(\nabla \mathbf{v}_h + \nabla \mathbf{v}_h^T), \\[2ex] \int_\Omega \sigma_h \cdot (\nabla \mathbf{w}_h + \nabla \mathbf{w}_h^T)/2 \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{w}_h \, dx + \int_\Omega \mathbf{g} \cdot \mathbf{w}_h \, da, \ \forall\, \mathbf{w}_h \in V_h. \end{cases}$$

In (2.11), the approximate deviatoric stress tensor $(\sigma_h + p_h\,\mathrm{Id})$ belongs to the space Y_h of piecewise constant symmetric matrix fields:

$$(2.12)\qquad Y_h = \{ D_h : \Omega \to \mathbb{R}^{N\times N}, D_h^T = D_h,\ D_h\big|_{\Omega_\ell^i} \in (P_o(\Omega_\ell^i))^{N\times N},\ \forall i = 1,2^N,\ \forall \ell = 1, N_h \}.$$

Proof: this theorem is the discrete equivalent of the existence Theorem 1.1. Up to (1.9), (1.10) its proof is identical, after replacement of $\mathcal{D}_1(\cdot)$ by $\mathcal{D}_1^e(\cdot)$, of K by K_h, of Y by Y_h and of X by

$$(2.13)\qquad X_h = \{w_h \in V_h,\ \int_\Omega q_h \operatorname{div} w_h = 0,\ \forall q_h \in P_h\}.$$

Now, since Y_h is made of piecewise constant matrix fields, (1.10) will also give

$$-(\sigma_D)_h \in \partial\, \mathcal{D}_1\, (D(v_h)) \quad \text{a.e. in } \Omega.$$

To finish the proof of (2.11), we introduce the operator B from V_h into P_h^* defined by

$$(2.14)\qquad \langle Bw_h, q_h\rangle = \int_\Omega q_h \operatorname{div} w_h\, dx,\ \forall q_h \in P_h,\ \forall w_h \in V_h,$$

whose Kernel is X_h, by definition. From the BREZZI inequality (2.2), (see for example GIRAULT-RAVIART[1979, p 41]), this operator is a continuous surjection from V_h onto P_h^*. Using the closed range theorem, its transpose is a one-to-one homeomorphism from P_h onto the orthogonal of X_h in V_h^*. But, from (1.9), the element $L_h(\cdot)$ of V_h^* defined by

$$(2.15)\qquad L_h(w_h) = \int_\Omega (\sigma_D)_h \cdot D(w_h)dx - \int_{\Gamma_2} g\cdot w_h\, da - \int_\Omega f\cdot w_h\, dx$$

belongs to this orthogonal subspace. Therefore, there exists a unique pressure field p_h in P_h such that

$$L_h(w_h) = \langle B^T p_h, w_h\rangle = \int_\Omega p_h \operatorname{div} w_h\, dx,\ \forall w_h \in V_h,$$

which is exactly (2.11). $\square$

2.3 Convergence result. In order to check that the discrete problem (2.10) is a good approximation of the continuous viscous flow problem (1.3) when the maximal diameter h of the triangulation $\mathcal{T}h$ goes to zero, one must study the behavior of the sequence (v_h) of solutions of (2.10) when h goes to zero. We will prove in this paragraph that (v_h) converges weakly towards solutions v of the continuous problem and

that the dissipated energy rate $J(v_h)$ converges towards $J(v)$. Moreover, under additional uniform convexity assumptions, such as those satisfied by Norton or by Bingham materials, there is strong convergence of (v_h) towards v in $W^{1,p}(\Omega)$. The next theorem summarizes these convergence properties, denoting by q the maximum of p (p is the exponent introduced in (1.6)) and 2 and by Y_p the space

$$Y_p = \{D \in (L^p(\Omega))^{N\times N},\ D^T = D\}.$$

THEOREM 2.2: Under the assumptions of Theorem 1.1, the sequence (v_h) of solutions of the discrete problem (2.10) decomposes itself into subsequences, each of them converging weakly in $W^{1,p}(\Omega)$ towards a solution v of the continuous incompressible viscous flow problem (1.3), when h goes to zero. The dissipated energy rate $J(v_h)$ also converges towards $J(v)$. Moreover, if the extended internal dissipation potential $\mathcal{D}_1^e$ is of the form

(2.16) $\quad \mathcal{D}_1^e(D) = \mathcal{D}_1\ (D) = G_o(D) + G_1(D),$

with G_1 convex and bounded below, G_o convex, differentiable and satisfying

$$(2.17)\begin{cases} (\|D\|_{o,p} + \|G\|_{o,p})^{q-p} \int_\Omega \left(\dfrac{\partial G_o}{\partial D}(G) - \dfrac{\partial G_o}{\partial D}(D)\right)\cdot(G-D)\,dx \geq C_o\|G-D\|^q_{o,p}, & \forall\{D,G\} \in Y_p^2, \\[2ex] \left|\int_\Omega\left(\dfrac{\partial G_o}{\partial D}(G) - \dfrac{\partial G_o}{\partial D}(D)\right)\cdot H dx\right| \leq C_4\|H\|_{o,p}\|G-D\|^{q-p+1}_{o,p}(\|D\|_{o,p}+\|G\|_{o,p})^{q-2}, & \forall\{D,G,H\} \in Y_p^3 \end{cases}$$

then the whole sequence (v_h) converges strongly in $W^{1,p}(\Omega)$ towards the unique solution v of the continuous problem (1.3).

Proof: The proof is an immediate generalization of the techniques used by GLOWINSKI-LIONS-TREMOLIERES [1981, p 361] in their study of Bingham fluids. It requires three steps. Step 1 (v_h) is bounded uniformly in h. Let v be a solution of the continuous problem (1.3) and let z_h be the element of K_h such that

(2.18) $\|v - z_h\|_{1,p} = \inf_{w_h \in K_h} \|v - w_h\|_{1,p}$.

From (2.6) (z_h) strongly converges towards v in $W^{1,p}(\Omega)$ as h goes to zero. Since, by extension, $J(\cdot)$ is continuous on $W^{1,p}(\Omega)$, this implies that, for h sufficiently small, we have

(2.19) $J(z_h) \leq J(v) + 1$.

But since v_h is a solution of (2.10), we get

$$J(v_h) \leq J(z_h) \leq J(v) + 1 .$$

From the convexity of $J(\cdot)$, this implies

(2.20) $J((v_h - \dot{u}_o)/2) \leq (J(-\dot{u}_o) + J(v) + 1)/2 = C_6$,

where, as usual, the notation C_i represents strictly positive numbers independent of x and h. From (2.9), $\mathcal{D}_1^e$ is coercive, thus (2.20) implies

$$\frac{C_1}{2^p} \int_\Omega |(\nabla(v_h - \dot{u}_o) + \nabla(v_h - \dot{u}_o)^T)/2|^p dx \leq C_6 + (\|f\| + \|g\|)\|v_h - \dot{u}_o\|_{1,p}$$

which, from the Korn's inequality and since p is strictly greater than 1, can only hold if

$$\|v_h\|_{1,p} \leq C_7 , \quad \forall h .$$

Step 2 weak convergence of (v_h). Since the sequence (v_h) is uniformly bounded in $W^{1,p}(\Omega)$, it decomposes itself into subsequences, each of them weakly converging in $W^{1,p}(\Omega)$. We still denote by (v_h) such a subsequence and denote by $\bar{v}$ its weak limit. Moreover, let v be a solution of (1.3) and let (z_h) be the sequence of elements of K_h defined by (2.18). Since v_h minimizes J over K_h, we have

(2.21) $J(v_h) \leq J(z_h)$, $\forall h$.

Going to the limit in (2.21) as h goes to zero, and using the weak lower semicontinuity of J on the left-hand side, the strong continuity of J on the right-hand side, we obtain

(2.22) $J(\bar{v}) \le \liminf J(v_h) \le \limsup J(v_h) \le \lim_{h\to 0} J(z_h) = J(v)$.

On the other hand, let q be any element of $L^{p^*}(\Omega)$ and let (q_h) be the sequence of elements of P_h which strongly approximates q in $L^{p^*}(\Omega)$. Since v_h belongs to K_h, we have

(2.23) $\int_\Omega q_h \operatorname{div} \bar{v}\, dx = \int_\Omega q_h \operatorname{div}(\bar{v}-v_h)dx \quad , \ \forall h.$

Going to the limit in (2.23) as h goes to zero and using the strong convergence of (q_h) and the weak convergence of (v_h) yields

$$\int_\Omega q \operatorname{div} \bar{v}\, dx = 0, \quad \forall q \in L^{p^*}(\Omega).$$

Moreover, from the weak continuity of the trace operator, $\bar{v} - \dot{u}_o$ has zero trace on Γ_1. So, finally, $\bar{v}$ belongs to K. But v minimizes J over K, therefore

$$J(v) \le J(\bar{v}).$$

Combined with (2.22), this implies that $J(\bar{v})$ is equal to $J(v)$ and that all inequalities in (2.22) are equalities. Therefore $\bar{v}$ is also a solution of (1.3) and the whole sequence $J(v_h)$ converges towards $J(v)$.

<u>Step 3</u> <u>Strong convergence of</u> (v_h). From now on, we suppose that the extended internal dissipation potential $\mathcal{D}_1^e$ satisfies (2.16) and (2.17). From (2.17), G_o is strictly convex on $\mathbb{R}_s^{N\times N}$, therefore, by addition, $\mathcal{D}_1^e$ is strictly convex. So is its restriction $\mathcal{D}_1$ on the space of symmetric matrices with zero trace. From Theorem 1.1, the solution v of (1.3) is then unique. Thus the only possible weak cluster point for the sequence (v_h) is v and the whole sequence (v_h) of solutions of (2.10) converges weakly towards v in $W^{1,p}(\Omega)$.

To prove its strong convergence, we first write the discrete weak equilibrium equations (2.11) and the continuous weak equilibrium equations (1.7) for the test function $w_h = w = z_h - v_h$ where z_h is the element of K_h defined in (2.18). This gives

$$
(2.24)\quad \begin{cases}
\int_\Omega \sigma_h \cdot D(\mathbf{z}_h - \mathbf{v}_h)\,dx = \int_\Omega \mathbf{f}\cdot(\mathbf{z}_h-\mathbf{v}_h)\,dx + \int_{\Gamma_2} \mathbf{g}\cdot(\mathbf{z}_h-\mathbf{v}_h)\,da, \\
\int_\Omega \sigma \cdot D(\mathbf{z}_h - \mathbf{v}_h)\,dx = \int_\Omega \mathbf{f}\cdot(\mathbf{z}_h-\mathbf{v}_h)\,dx + \int_{\Gamma_2} \mathbf{g}\cdot(\mathbf{z}_h-\mathbf{v}_h)\,da, \\
(\sigma + p\,\mathbf{Id}) \in \partial\, \mathcal{D}_1(\dot{\mathbf{E}}), \quad \dot{\mathbf{E}} = \tfrac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T), \\
(\sigma_h + p_h\,\mathbf{Id}) \in \partial\, \mathcal{D}_1^e(\dot{\mathbf{E}}_h), \quad \dot{\mathbf{E}}_h = \tfrac{1}{2}(\nabla\mathbf{v}_h+\nabla\mathbf{v}_h^T),
\end{cases}
$$

where the notation $D(\mathbf{w})$ represents as usual the symmetric component $(\nabla\mathbf{w}+\nabla\mathbf{w}^T)/2$ of the matrix $\nabla\mathbf{w}$ in $\mathbb{R}^{N\times N}$. By definition of the subgradient, the third line of (2.24) is equivalent to

$$
(2.25)\quad (\sigma+p\,\mathbf{Id})\cdot\mathbf{H} \le \mathcal{D}_1(\dot{\mathbf{E}}+\mathbf{H}) - \mathcal{D}_1(\dot{\mathbf{E}}),\ \forall\, \mathbf{H} \in \mathbb{R}_s^{N\times N} \ \text{ with } \ \mathrm{Tr}(\mathbf{H}) = 0.
$$

Since $(\sigma+p\,\mathbf{Id})$ has zero trace and since $\mathcal{D}_1^e$ is an extension of $\mathcal{D}_1$ which satisfies (2.9), (2.24) yields

$$
\begin{aligned}
(\sigma+p\,\mathbf{Id})\cdot(\mathbf{H}+q\,\mathbf{Id}) = (\sigma+p\,\mathbf{Id})\cdot\mathbf{H} &\le \mathcal{D}_1(\dot{\mathbf{E}}+\mathbf{H}) - \mathcal{D}_1(\dot{\mathbf{E}}) \\
&\le \mathcal{D}_1^e(\dot{\mathbf{E}}+\mathbf{H}+q\,\mathbf{Id}) - \mathcal{D}_1^e(\dot{\mathbf{E}}),\ \forall\, q \in \mathbb{R},
\end{aligned}
$$

or, in other words

$$
(2.26)\quad (\sigma+p\,\mathbf{Id}) \in \partial\,\mathcal{D}_1^e(\dot{\mathbf{E}})\ .
$$

Now setting

$$
t = \sigma + p\,\mathbf{Id} - \frac{\partial G_o}{\partial D}(\dot{\mathbf{E}}), \quad t_h = \sigma_h + p_h\,\mathbf{Id} - \frac{\partial G_o}{\partial D}(\dot{\mathbf{E}}_h)\ ,
$$

we obtain by substraction from (2.24) and (2.26)

$$(2.27)\quad \begin{cases} \int_\Omega (t-t_h)\cdot D(z_h-v_h)\,dx = \int_\Omega \left(\frac{\partial G_o}{\partial D}(\dot{E}_h) - \frac{\partial G_o}{\partial D}(\dot{E})\right)\cdot D(z_h-v_h)\,dx \\ \qquad\qquad + \int_\Omega (p-p_h)\,\mathrm{div}\,(z_h-v_h)\,dx\,, \\ t \in \partial G_1(\dot{E})\,,\quad t_h \in \partial G_1(\dot{E}_h)\,. \end{cases}$$

But, since t and t_h are subdifferentials of G_1, we have by definition

$$(2.28)\quad \begin{cases} \int_\Omega t\cdot D(v_h-v)\,dx \le \int_\Omega \{G_1(\dot{E}_h)-G_1(\dot{E})\}\,dx\,. \\ \int_\Omega t_h\cdot D(z_h-v_h)\,dx \le \int_\Omega \{G_1(D(z_h))-G_1(\dot{E}_h)\}\,dx\,. \end{cases}$$

A suitable combination of (2.27) and (2.28) yields

$$(2.29)\quad \int_\Omega \left(\frac{\partial G_o}{\partial D}(D(z_h)) - \frac{\partial G_o}{\partial D}(\dot{E}_h)\right)\cdot D(z_h-v_h)\,dx \le \int_\Omega \left(\frac{\partial G_o}{\partial D}(D(z_h)) - \frac{\partial G_o}{\partial D}(\dot{E})\right)\cdot D\,(z_h-v_h)\,dx$$

$$+ \int_\Omega (p-p_h)\,\mathrm{div}\,(z_h-v_h)\,dx + \int_\Omega \{t\cdot D(v-z_h) + G_1(D(z_h)) - G_1(\dot{E})\}\,dx\,.$$

Both v_h and z_h are elements of K_h, so we can replace in (2.29)

$$\int_\Omega (p-p_h)\,\mathrm{div}\,(z_h-v_h)\,dx \quad \text{by} \quad \int_\Omega (p-q_h)\,\mathrm{div}\,(z_h-v_h)\,dx\,,$$

where q_h is the element of P_h which approximates p in $L^{p^*}(\Omega)$. Once this replacement done, we have from (2.29), (2.17) and the Korn's inequality

$$(2.30)\quad C_{10}(\|z_h\|_{1,p}+\|v_h\|_{1,p})^{p-q}\,\|z_h-v_h\|_{1,p}^q \le C_{11}\|z_h-v_h\|_{1,p}\|z_h-v\|_{1,p}^{p-q+1}(\|z_h\|_{1,p} + \|v\|_{1,p})^{q-2}$$

$$+ \|p-q_h\|_{o,p^*}\|z_h-v_h\|_{1,p} + C_{12}\left|\int_\Omega \{t\cdot D(v-z_h) + G_1(D(z_h)) - G_1(\dot{E})\}\,dx\right|\,.$$

Since by construction z_h and q_h converge strongly respectively towards v and p in $W^{1,p}(\Omega)$ and $L^{p^*}(\Omega)$, since from Step 2 (v_h) is uniformly bounded in $W^{1,p}(\Omega)$ and since, from (2.9), the integral of G_1 is continuous on Y_p, the right-hand side of (2.30) converge towards 0 when h goes to zero. Therefore $\|z_h - v_h\|_{1,p}$ must also converge to zero, and from the triangular inequality, the sequence v_h <u>strongly converges towards</u> v in $W^{1,p}(\Omega)$ when h goes to zero. □

<u>REMARK 2.2</u>: Three facts are crucial in our proof of convergence: the existence of an approximate pressure p_h, the existence of a sequence (z_h) of elements of K_h approximating v and the existence of a sequence (q_h) in P_h approximating any element q of $L^{p^*}(\Omega)$. Although not necessary the BREZZI condition (2.2) is a basic tool for proving the first two facts. □

<u>REMARK 2.3</u>: Norton and Bingham materials satisfy the uniform convexity assumptions (2.16) and (2.17) (SCHEURER [1977], *GLOWINSKI-MAROCCO* [1975]) and therefore, strong convergence can be proved in both cases. Moreover, the speed of convergence of (v_h) towards v can easily be estimated by (2.30) as a function of the quantity $\left(\inf_{w_h \in K_h} \|w_h - v\|_{1,p}\right)^{1/q}$, using the Lipschitz continuity of G_1 and the identity

$$ab \le \frac{a^q}{q} + \frac{b^{q^*}}{q^*} .$$

In addition, since the dissipation potential is continuously differentiable for Norton materials, the strong convergence of (v_h) implies in this case the strong convergence of the discrete stresses (σ_h) towards (σ) in $(L^{p^*}(\Omega))^{N\times N}$. □

3 AUGMENTED LAGRANGIANS

3.1 Formulation of the discrete problems as saddle-point problems.

In view of the numerical solution of the approximate viscous flow problem (2.10) by augmented lagrangian techniques, we must first reformulate (2.10) under a slightly different form.

To do that, observe that, if we replace $\dot{\mathbf{u}}_o$ by its $H_o^1(\Omega)$ projection over the space of continuous functions whose restriction to each subelement Ω_ℓ^i is a first degree polynomial, we can rewrite (2.10) as

(3.1) Minimize $F(\mathbf{D}(\mathbf{w}_h)) + G(\mathbf{w}_h)$ over K_h, with

(3.2) $$\mathbf{D}(\mathbf{w}_h) = \tfrac{1}{2}(\nabla \mathbf{w}_h + \nabla \mathbf{w}_h^T),$$

(3.3) $$\begin{cases} F : \quad Y_h \to \mathbb{R}, \\ F(\mathbf{G}_h) = \int_\Omega D_1^e(\mathbf{G}_h)\, dx, \end{cases}$$

(3.4) $$\begin{cases} G : \quad K_h \to \mathbb{R}, \\ G(\mathbf{w}_h) = -\int_\Omega \mathbf{f}\cdot\mathbf{w}_h\, dx - \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{w}_h\, da, \end{cases}$$

(3.5) $$Y_h = \{\mathbf{D}_h : \bar{\Omega} \to \mathbb{R}^{N\times N},\ \mathbf{D}_h^T = \mathbf{D}_h,\ \mathbf{D}_h|_{\Omega_\ell^i} \in (P_o(\Omega_\ell^i))^{N\times N},\ \forall\, i = 1, 2^N,\ \forall \ell = 1, N_h\}.$$

If we follow the methodology of FORTIN-GLOWINSKI [1982], we can then replace (3.1) by its augmented lagrangian formulation

(3.6) $$\begin{cases} \underline{\text{Find a saddle-point}}\ \ \{\{\mathbf{v}_h, \mathbf{H}_h\}, \lambda_h\}\ \underline{\text{of the augmented lagrangian}} \\ L_R(\mathbf{w}_h, \mathbf{G}_h, \mu_h) = F(\mathbf{G}_h) + G(\mathbf{w}_h) + \frac{R}{2}\|\mathbf{D}(\mathbf{w}_h) - \mathbf{G}_h\|_{0,2}^2 - \langle \mu_h, \mathbf{D}(\mathbf{w}_h) - \mathbf{G}_h \rangle \\ \underline{\text{over the set}}\ \ (K_h \times Y_h) \times Y_h, \end{cases}$$

where R is any positive number and where $\langle\cdot,\cdot\rangle$ denotes the classical $L^2(\Omega)$ scalar product over $(L^2(\Omega))^{N\times N}$. Observe that (3.6) imposes the incompressibility condition on the continuous variable $\mathbf{v}_h$ ($\mathbf{v}_h$ must belong to the set K_h of approximately incompressible velocity fields) but minimizes the nonlinear functional $F(\cdot)$ with respect to the piecewise constant variable $\mathbf{G}_h$. In other words, there is a splitting of the difficulties of our problem (nonlinearity and incompressibility) between these two variables.

THEOREM 3.1: The augmented lagrangian problem (3.6) and the approximate incompressible viscous flow problem (2.10) are equivalent: to any solution $\mathbf{v}_h$ of (2.10), one can associate a solution $\{(\mathbf{v}_h,\mathbf{E}_h),\lambda_h\}$ of (3.6) and conversely. Moreover, $\mathbf{E}_h$ is equal to $D(\mathbf{v}_h)$ and there exists an approximate pressure field p_h in P_h such that the approximate stress tensor field $(-\lambda_h - p_h\,\mathrm{Id})$ satisfies the discrete equilibrium equations and constitutive laws (2.11).

Proof: First, let $\mathbf{v}_h$ be a solution of (2.10) and let σ_h and p_h be the associated discrete stress and *pressure fields*. From (2.11), Theorem 2.1, we have

$$(3.7)\quad \begin{cases} \int_\Omega (\sigma_h + p_h\mathrm{Id})\cdot\mathbf{G}_h\,d\mathbf{x} \le \int_\Omega \{\mathcal{D}_1^e(\dot{\mathbf{E}}_h+\mathbf{G}_h) - \mathcal{D}_1^e(\dot{\mathbf{E}}_h)\}\,d\mathbf{x},\ \forall\,\mathbf{G}_h \in Y_h, \\ \int_\Omega \sigma_h\cdot D(\mathbf{w}_h)\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{w}_h\,d\mathbf{x} + \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{w}_h\,da,\ \forall\,\mathbf{w}_h \in V_h. \end{cases}$$

In particular, taking $\mathbf{w}_h$ as $(\mathbf{z}_h-\mathbf{v}_h)$ where $\mathbf{z}_h$ is any element of K_h, and since, by construction of K_h, div $(\mathbf{z}_h-\mathbf{v}_h)$ is equal to zero in the dual of P_h, we have

$$\int_\Omega (\sigma_h+p_h\,\mathrm{Id})\cdot D(\mathbf{z}_h-\mathbf{v}_h)\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot(\mathbf{z}_h-\mathbf{v}_h)\,d\mathbf{x} + \int_{\Gamma_2} \mathbf{g}\cdot(\mathbf{z}_h-\mathbf{v}_h)\,da,\ \forall\,\mathbf{z}_h \in K_h.$$

Substracting this to the first inequality of (3.7) yields

$$L_R(v_h,\dot{E}_h, -\sigma_h - p_h Id) \leq L_R(z_h,\dot{E}_h + G_h, -\sigma_h - p_h Id) - \frac{R}{2} |D(z_h) - \dot{E}_h - G_h|^2_{0,2}$$

for any $\{z_h, G_h\}$ in $K_h \times Y_h$. Since in addition

$$L_R(v_h,\dot{E}_h,-\sigma_h - p_h Id) = L_R(v_h,\dot{E}_h,\mu_h) = J(v_h), \ \forall \mu_h \in Y_h,$$

$\{\{v_h,\dot{E}_h\}, -\sigma_h - p_h Id\}$ is indeed a <u>saddle-point of the augmented lagrangian</u> $L_R(\cdot,\cdot,\cdot)$ over $(K_h \times Y_h) \times Y_h$.

Conversely, let $\{\{v_h, E_h\}, \lambda_h\}$ be a solution of the augmented lagrangian problem (3.6). Then, we must have

$$L_R(v_h, E_h, \lambda_h) \geq L_R(v_h, E_h, \mu_h), \ \forall \mu_h \in Y_h$$

which can only hold if we have

(3.8) $\quad E_h = D(v_h)$.

Taking (3.8) into account, the second saddle-point inequality yields

(3.9) $\quad L_R(v_h, D(v_h), \lambda_h) \leq L_R(w_h, G_h, \lambda_h), \ \forall \{w_h, G_h\} \in K_h \times Y_h.$

In particular, by taking G_h as $D(w_h)$, (3.9) implies

$$J(v_h) \leq J(w_h) \ , \ \forall w_h \in K_h,$$

and v_h is indeed a <u>solution of the original minimization problem</u> (2.10).

To further characterize any solution $\{\{v_h, E_h\}, \lambda_h\}$ of the augmented lagrangian problem (3.6), we again use (3.8) and (3.9). From (3.8), E_h is <u>necessarily equal</u> to $D(v_h)$. On the other hand, introducing the space X_h defined in (2.13), (3.9) can be rewritten as

$$L_R(\mathbf{v}_h, D(\mathbf{v}_h), \lambda_h) \leq L_R(\mathbf{v}_h+\mathbf{w}_h, D(\mathbf{v}_h) + G_h, \lambda_h), \ \forall \{\mathbf{w}_h, G_h\} \in X_h \times Y_h,$$

Equivalently, if we consider $L_R(\mathbf{v}_h+\cdot, D(\mathbf{v}_h)+\cdot, \lambda_h)$ as a convex function of the pair $\{\mathbf{w}_h, G_h\}$ on the space $X_h \times Y_h$, we can write (3.9) as

$$\{0,0\} \in \partial L_R(\mathbf{v}_h+0, D(\mathbf{v}_h)+0, \lambda_h) \quad \text{in} \quad X_h^* \times Y_h^* .$$

A direct calculation characterizes the elements of this subgradient as the pairs $\{\mathbf{g}_h, \mu_h\}$ of $X_h \times Y_h$ such that

$$(3.10)\quad \begin{cases} \int_\Omega D(\mathbf{g}_h)\cdot D(\mathbf{w}_h)dx \leq -\int_\Omega \lambda_h\cdot D(\mathbf{w}_h)dx - \int_\Omega \mathbf{f}\cdot\mathbf{w}_h dx - \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{w}_h da, \ \forall \mathbf{w}_h \in X_h, \\ \\ \int_\Omega \mu_h\cdot G_h dx \leq \int_\Omega \{\mathcal{D}_1^e(D(\mathbf{v}_h) + G_h) - \mathcal{D}_1^e(D(\mathbf{v}_h)) + \lambda_h\cdot G_h\}dx, \ \forall G_h \in Y_h. \end{cases}$$

Setting $\mathbf{g}_h$ and μ_h to zero in (3.10), we simply obtain the variational system (1.9)-(1.11) with $(\sigma_D)_h = -\lambda_h$. As seen in the proof of Theorem 2.1, this in turn implies (2.11) with $\sigma_h = -\lambda_h - p_h \, \mathrm{Id}$, and our proof is complete. □

<u>REMARK 3.1</u>: In order to accelerate the convergence of the algorithm to be used for the solution of the augmented lagrangian problem (3.6), it is usually better to replace, in the definition of the augmented lagrangian, the classical $L^2(\Omega)$ scalar product by an equivalent weighted scalar product of the type

$$\langle C, D \rangle = \int_\Omega r(x)\, C \cdot D\, dx.$$

Here $r(x)$ is a strictly positive scalar function of $L^\infty(\Omega)$, bounded away from zero, which can be arbitrarily chosen. Proper choices for this function will be discussed later. With this new scalar product, $L_R(\cdot,\cdot,\cdot)$ becomes

$$L_R(\mathbf{w}_h, G_h, \mu_h) = \int_\Omega \mathcal{D}_1^e(G_h)\, dx - \int_\Omega \mathbf{f}\cdot\mathbf{w}_h\, dx - \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{w}_h\, da$$

$$+ \frac{R}{2} \int_\Omega r(x)|G_h - D(\mathbf{w}_h)|^2\, dx - \int_\Omega r(x)\mu_h\cdot(D(\mathbf{w}_h) - G_h)dx . \quad □$$

3.2 Numerical algorithm The fundamental interest of the equivalent augmented lagrangian formulation (3.6) is the existence of a very cheap and simple algorithm for its numerical solution. This algorithm combines an Uzawa algorithm for the solution of the saddle-point problem and a block-relaxation technique for the solution of the minimization problems associated to the primal variable $\{w_h, G_h\}$. Dropping the subscript h from all variables for simplicity, this algorithm is

(3.11) Let $\{\lambda^o, H^{-1}\}$ be given in $Y_h \times K_h$,

Then, for $n > 0$, H^{n-1} and λ^n being known, we compute $\{v^n, H^n\}$ in $K_h \times Y_h$ by block-relation, i.e. by setting

$$H_o^n = H^{n-1},$$

and by computing sequentially v_k^n and H_k^n by solving

$$(3.12)\quad L_R(v_k^n, H_{k-1}^n, \lambda^n) \le L_R(w, H_{k-1}^n, \lambda^n),\ \forall\, w \in K_h,$$

$$(3.13)\quad L_R(v_k^n, H_k^n, \lambda^n) \le L_R(v_k^n, G, \lambda^n),\ \forall\, G \in Y_h;$$

Once $\{v^n, H^n\}$ is known, the Lagrange multiplier λ is updated by

$$(3.14)\quad \lambda^{n+1} = \lambda^n - R(D(v^n) - H^n).$$

Many variants exist for this algorithm and are described for example in FORTIN-GLOWINSKI [1982]. Usually, the block-relaxation (i.e. the loop (3.12)-(3.13) on k) is only carried out for one to five iterations.

Observe that the above algorithm only considers one variable at a time and therefore takes full advantage of the splitting of the difficulties achieved by the saddle-point

formulation (3.6). First, in (3.12), the matrix field $\mathbb{H}^n{}_{k-1}$ and the multiplier λ^n are supposed to be known, and the algorithm minimizes the augmented lagrangian L_R with respect to the velocity field $\mathbf{w}$ in K_h. As function of the velocity field, L_R is quadratic and corresponds to the energy dissipated by an incompressible Stokesian fluid, flowing viscously under the action of the external loads $\{f,g\}$. In other words, (3.12) is a classical linear stationary Stokes problem, discretized by mixed finite element methods. Many numerical techniques are available for its solution, and we refer to TAYLOR-HOOD [1973], GIRAULT-RAVIART [1979] or GLOWINSKI-PIRONNEAU [1979] for the practical description of such techniques. In our numerical experiments, we will choose a conjugate gradient method operating on the hydrostatic pressure space P_h, which only requires the inversion of sparse, fixed, positive definite, symmetric matrices and therefore only uses little computer running time and memory core (FORTIN-GLOWINSKI [1982 p57]). In any case, most finite element codes now propose efficient subroutines for the solution of the Stokes problem, which can be blindly used for solving (3.12).

Then, the algorithm supposes the velocity field $\mathbf{v}_k^n$ and the multiplier λ^n given, and in (3.13) minimizes L_R with respect to the matrix field G in Y_h. The incompressibility condition and the spatial derivatives of G are not involved in (3.13): this is an unconstrained local convex minimization problem whose numerical solution, described in details in the next section, reduces to the solution in parallel of independent convex minimization problems set on $\mathbb{R}^N (N=2 \text{ or } 3)$.

Finally, after a few resolutions of (3.12) and (3.13), the algorithm updates the multiplier λ^n by the explicit formula (3.14), so that the constraint $D(\mathbf{v}^n) = \mathbb{H}^n$ can be better satisfied by the solution of (3.12)-(3.13), and then returns to (3.12) and (3.13).

3.3 Convergence of the algorithm (3.11)-(3.14). We now study the convergence properties of the above Uzawa algorithm, considering the basic particular case where only one iteration of block-relaxation is done at each step of the Uzawa algorithm. In our

study, it will be most important to work on Y_h with the precise weighted $L^2(\Omega)$ norm which is used in the construction of the augmented lagrangian L_R (see Remark 3.1).

Then, if we denote by $\{v,H,\lambda\}$ the solution of the augmented lagrangian problem (3.6), by λ^n the multiplier calculated in (3.14), by H^n the matrix field H_1^n calculated in (3.13) and by v^n the vector field v_1^n calculated in (3.12), we can prove

CONVERGENCE THEOREM 3.2: Under the assumption of the existence Theorem 1.1 ($\mathcal{D}_1$ convex, continuous, coercive), the sequence $\{\lambda^n\}$ is bounded in Y_h, the difference $(D(v^n) - H^n)$ converges to zero in Y_h, and the quantity $F(H^n) + G(v^n)$ converges towards the dissipated energy rate $J(v)$. If in addition (2.16) is satisfied together with the first line of (2.17) ($\mathcal{D}_1(\cdot)$ uniformly convex on the bounded sets of Y_h), then the sequence $\{v^n,H^n\}$ converges towards $\{v,H\}$ strongly in $K_h \times Y_h$. Finally, if the internal dissipation potential $\mathcal{D}_1(\cdot)$ is continuously differentiable, and if its gradient is invertible with a coercive and Lipschitz continuous inverse, that is if $\mathcal{D}_1(\cdot)$ satisfies

$$
(3.15)\quad \begin{cases} \int_\Omega r(x)\ |G_1 - G_2|^2 dx \le C_{13}^2 \int_\Omega r(x)\ |\partial\mathcal{D}_1(G_1) - \partial\mathcal{D}_1(G_2)|^2\ dx, \\[2ex] \int_\Omega (\partial\mathcal{D}_1(G_1) - \partial\mathcal{D}_1(G_2))\cdot(G_1-G_2)dx \ge C_{14} \int_\Omega r(x)|\partial\mathcal{D}_1(G_1) - \partial\mathcal{D}_1(G_2)|^2 dx, \end{cases}
$$

for any G_1 and G_2 in Y_h, then we can prove that the sequence $\{v^n,H^n,\lambda^n\}$ converges linearly towards $\{v,H,\lambda\}$ in $K_h \times Y_h \times Y_h$ with an asymptotic constant bounded by

$$
C_{15} = (1-2RC_{14}/(1+C_{13}R)^2)^{1/2}.
$$

Proof: Since $\{v,H,\lambda\}$ is a solution of the saddle-point problem (3.6), the following extremality relation is satisfied:

(3.16) $\quad F(H^n) + G(v^n) + \langle \lambda, H^n - D(v^n) \rangle \geq F(H) + G(v).$

Moreover, by construction, the solutions v^n and H^n of (3.12) and (3.13) satisfy the extremality relations

$$G(w) - G(v^n) + \langle R(D(v^n) - H^{n-1}) - \lambda^n, D(w-v^n) \rangle \geq 0, \ \forall\, w \in K_h,$$

$$F(G) - F(H^n) + \langle R\,(H^n-D(v^n)) + \lambda^n, H - H^n \rangle \geq 0, \ \forall\, G \in Y_h,$$

respectively. By addition, setting $w = v$, and $G = H$, we get:

$$F(H) + G(v) - R\|D(v^n) - H^n\|^2 - R \langle H^{n-1} - H^n, D(v-v^n) \rangle \tag{3.17}$$
$$+ \langle \lambda^n, D(v^n) - H^n \rangle \geq F(H^n) + G(v^n) .$$

Adding (3.17) to (3.16), we then obtain

$$- R\|D(v^n) - H^n\|^2 - R \langle H^{n-1} - H^n, D(v-v^n) \rangle + \langle \lambda^n-\lambda, D(v^n) - H^n \rangle \geq 0. \tag{3.18}$$

Combining (3.18) with the construction (3.14) of λ^{n+1} finally yields

$$\|\lambda^n - \lambda\|^2 - \|\lambda^{n+1} - \lambda\|^2 \geq R^2 \|D(v^n) - H^n\|^2 + 2R^2 \langle H^{n-1} - H^n, D(v-v^n) \rangle . \tag{3.19}$$

On the other hand, using (3.13) and (3.14) at iteration (n-1), we can estimate the right-hand side of (3.19) by standard algebraic manipulations. Exactly as in FORTIN-GLOWINSKI [1982 p117, equations (5.17) to (5.24)], setting $F_o = 0$, $\rho = R$ and inverting the sign of λ, we have the following estimate

$$2R^2 \langle D(v^n-v), H^n - H^{n-1} \rangle \geq R^2(\|H^n-H\|^2-\|H^{n-1}-H\|^2) + R^2\|H^n - H^{n-1}\|^2,$$

which, combined with (3.19), gives

(3.20) $\quad (|\lambda^n - \lambda|^2 + R^2|\mathbb{H}^{n-1} - \mathbb{H}|^2) - (|\lambda^{n+1} - \lambda|^2 + R^2|\mathbb{H}^n - \mathbb{H}|^2) \geq$

$$R^2| D(\mathbf{v}^n) - \mathbb{H}^n|^2 + R^2|\mathbb{H}^n - \mathbb{H}^{n-1}|^2 .$$

The positive sequence $|\lambda^n - \lambda|^2 + R^2|\mathbb{H}^{n-1} - \mathbb{H}|^2$ is therefore decreasing and thus converges to a limit. This implies that the right-hand side of (3.20) must converge to zero and we finally obtain

$$|\lambda^n - \lambda|^2 + R^2|\mathbb{H}^{n-1} - \mathbb{H}|^2 \text{ is bounded,}$$

$$\lim_{n\to+\infty} |D(\mathbf{v}^n) - \mathbb{H}^n|^2 = 0,$$

$$\lim_{n\to+\infty} |\mathbb{H}^n - \mathbb{H}^{n-1}| = 0 .$$

These convergence results, used back in (3.16) and (3.17) obviously imply the convergence of $F(\mathbb{H}^n) + G(\mathbf{v}^n)$ towards $F(\mathbb{H}) + G(\mathbf{v})$.

Now, if $\mathcal{D}_1(\cdot)$ is uniformly convex on the bounded sets of Y_h, the convergence of the energy rate and the boundedness of $\mathbb{H}^n$ imply the convergence of the arguments $\mathbb{H}^n$ and $\mathbf{v}^n$ respectively towards $\mathbb{H}$ and $\mathbf{v}$.

Finally, since, from the Korn's inequality, $\mathbb{B}^T\mathbb{B}$ is an isomorphism from K_h onto its dual, we can prove the linear convergence of the sequence $\{\mathbf{v}^n, \mathbb{H}^n, \lambda^n\}$ by applying a result of LIONS-MERCIER [1979, Prop. 4, p 970] which will be applicable here as soon as (3.15) is satisfied(see FORTIN-GLOWINSKI [1982, p 300] for more details). □

<u>REMARK 3.1</u>: Condition (3.15) is satisfied at least locally for Norton materials. It is not satisfied in the general case, but linear convergence of the sequence $\{\mathbf{v}^n, \mathbb{H}^n\}$ can still be observed numerically in almost any case. □

<u>REMARK 3.2</u>: The asymptotic constant C_{15} appears numerically not to be optimal. <u>Nevertheless, its expression as a function of</u> C_{13} <u>and</u> C_{14} <u>will indicate the right</u>

strategy to follow for the choice of the parameter R and of the weight $r(x)$. Since C_{13} and C_{14} highly depend on the weight $r(x)$ used in the definition of the augmented lagrangian L_R, C_{15} is a function of R and of r. The right strategy for the choice of R and of r now consists in trying to keep C_{15} as small as possible. By choosing R close to $1/C_{13}$, C_{15} becomes approximately equal to $C_{16} = (1-C_{14}/2C_{13})^{1/2}$. By taking the weight $r(x)$ so that the products

(3.21) $\int_\Omega (\partial \mathcal{D}_1(\mathbf{H}_1) - \partial \mathcal{D}_1(\mathbf{H}_2)) \cdot (\mathbf{G}_1-\mathbf{G}_2)dx$ and (3.22) $\int_\Omega r(x)(\mathbf{G}_1-\mathbf{G}_2)\cdot(\mathbf{H}_1-\mathbf{H}_2)dx$

remain close to each other when $\mathbf{H}_1$ and $\mathbf{H}_2$ are in the neighborhood of $\mathbf{H}$, the ratio C_{14}/C_{13} gets close to 1 and C_{15} reaches the value $1/\sqrt{2}$. The final strategy for choosing R and r is therefore:

(i) choose $r(x)$ by matching (3.21) and (3.22);

(ii) take R close to $1/C_{13}$, which will usually be close to 1 if $r(\cdot)$ is properly chosen.

REMARK 3.3: If $\mathcal{D}_1$ is quadratic and if we have equality between (3.21) and (3.22), then for $R = 1$, $\mathbf{v}^n$ converges in 2 iterations and $\mathbf{H}^n$ converges linearly with asymptotic constant .5 (FORTIN-GLOWINSKI [1982, p 119]). In other cases, with proper choices of R and of r, we usually observe linear convergence of $(\mathbf{v}^n, \mathbf{H}^n)$ with an asymptotic constant around .7. □

REMARK 3.4: Linear convergence compares unfavorably to the quadratic convergence expected for conjugate gradient or for Newton algorithms. But Newton method requires $\mathcal{D}_1$ to be twice differentiable, the factorization of quite a few finite element matrices, and its convergence rate can be very slow for weakly convex dissipation potentials. In general, it is not a good method for solving (2.10). On the other hand, a conjugate gradient method with preconditioning, of the type

- take u_o in K_h;
- solve $< D(g_o), D(w) > = < J'(u_o), w >$, $\forall\, w \in K_h$;
- set $z_o = g_o$;
- for $n = 0$, until satisfied do

  $\rho_n = \text{Arg Min } J(u_n - \rho z_n)$,

  $u_{n+1} = u_n - \rho_n z_n$,

  solve $< D(g_{n+1}), D(w) > = < J'(u_{n+1}), w >$, $\forall\, w \in K_h$,

  $\gamma_n = < D(g_{n+1}), D(g_{n+1} - g_n) > / < D(g_n), g_n >$,

  $z_{n+1} = g_{n+1} + \gamma_n z_n$,

  end loop on n ;

where $<\cdot, \cdot>$ is an adequate weighted $L^2(\Omega)$ scalar product on Y_h, will only be efficient if $\mathcal{D}_1(\cdot)$ is differentiable, if the scalar product on Y_h is correctly chosen and if a very efficient Stokes solver is available for computing g_{n+1}. If this is the case, the conjugate gradient method will be twice as fast as the Uzawa algorithm (3.11)-(3.14). If this is not the case, Algorithm (3.11)-(3.14) appears to be one of the only reasonable numerical method for solving (2.10).

4. THE PROBLEMS IN DEFORMATION RATES.

4.1 The local problems. Problem (3.13) appears as one step of the algorithm proposed herein for the numerical solution of the viscous flow problems in quasistatic viscoplasticity, once these problems have been approximated by simplicial finite elements of order one and decomposed under an augmented lagrangian form. Recall that here, this problem consists in

(3.13) Minimizing $L_R(\mathbf{v},\cdot,\lambda)$ over Y_h

with

$$L_r(\mathbf{v},G,\lambda) = \int_\Omega \mathcal{D}_1^e(G)\,dx - \int_\Omega f\cdot v\,dx - \int_{\Gamma_2} g\cdot v\,da$$

$$+\frac{R}{2}\int_\Omega r(x)\,|G-(\nabla v+\nabla v^T)/2|^2 dx - \int_\Omega r(x)\lambda\cdot\left[(\nabla v+\nabla v^T)/2 - G\right] dx,$$

$$Y_h = \{D:\ \Omega \to R_s^{N\times N},\ D|_{\Omega_\ell^i} \in (P_o(\Omega_\ell^i))^{N\times N},\ \forall i = 1, 2^N,\ \forall \ell = 1, N_h\},$$

and that it is the only nonstandard step in this algorithm, the other steps consisting of linear Stokes problems and explicit variables updating, respectively.

Since all the elements of Y_h are matrix fields which are constant on each Ω_ℓ^i, and since the functional L_R does not involve any distributional derivative of G, Problem (3.13) can equivalently be written as

(4.1) $\forall\ i = 1, 2^N,\ \forall\ \ell = 1, N_h$, Minimize $J_\ell^i(G)$ over $R_s^{N\times N}$,

with

(4.2) $$J_\ell^i(G) = \mathcal{D}_1^e(G) + \frac{rR}{2}|G|^2 - rG\cdot(R(\nabla v+\nabla v^T)/2 - \lambda)|_{\Omega_\ell^i}\ .$$

Here, we are simply using the fact that the minimum value of the sum of independent terms is equal to the sum of the minimum value of each term. Then, Problem (3.13) reduces to

the solution in parallel of $N_h x 2^N$ local independent convex minimization problems set on $\mathbb{R}_s^{N\times N}$ (N = 2 or 3).

Using a general purpose minimization algorithm for the solution of each local problem (4.1) is not adviseable here for two main reasons:

(i) such an algorithm is very difficult to implement because it must be able to handle general nondifferentiable convex dissipation potentials $\mathcal{D}_1$;

(ii) such an algorithm is usually expensive in computer running time.

An easier and more efficient strategy consists in adapting each time the minimization algorithm to the specific class of potentials $\mathcal{D}_1$ which is under consideration. Doing that, we have most of the times been able to reduce each local problem (4.1) to a one-dimensional convex minimization problem set on $\mathbb{R}_+$. The remainder of this report will describe the derivation of such efficient numerical techniques in the case of Norton materials, of Bingham materials, and of Tresca type materials in plane stresses, respectively. But before, we will derive a very useful simplification of the local problem (4.1).

4.2 Reduction of the local problems. We begin by recalling several well-known results of matrix theory, which will enable us to reduce the local problems (4.1) which are set on $\mathbb{R}_s^{N\times N}$ to local convex minimization problems set on $\mathbb{R}^N$, (N = 2 or 3).

Lemma 4.1 (VON NEUMANN[1937]). Let A and B be two matrices in $\mathbb{R}^{N\times N}$ with singular values $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N \geq 0$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0$. Then

$$\mathrm{Tr}(A\,B) \leq \sum_{i=1}^{N} \alpha_i \beta_i \; .$$

Lemma 4.2. Let A and B be two symmetric matrices in $R_s^{N\times N}$ with eigenvalues $A_1 > A_2 > \cdots > A_N$ and $B_1 > B_2 > \cdots > B_N$. Then

$$A \cdot B = \mathrm{Tr}\ (A\ B) < \sum_{i=1}^{N} A_i\ B_i\ .$$

Proof: The result follows from the decomposition

$$\mathrm{Tr}\ (A\ B) = \mathrm{Tr}[(A-A_N\ \mathrm{Id})(B-B_N\ \mathrm{Id})] + A_N \mathrm{Tr}(B) + B_N \mathrm{Tr}(A) - N\ A_N B_N$$

and from the application of Lemma 4.1 to the first term of the right-hand side. □

Lemma 4.3 Let D^d be a diagonal matrix with diagonal terms $D_1 > D_2 > \cdots > D_N$ and A be a symmetric matrix with eigenvalues $A_1 > A_2 > \cdots > A_N$. Then

$$\underset{PP^T=\mathrm{Id}}{\mathrm{Max}} \left[\mathrm{Tr}[P^T D^d\ P\ A]\right] = \mathrm{Tr}\ [Q\ D^d\ Q^T A] = \sum_{i=1}^{N} D_i A_i ,$$

where Q is an orthogonal matrix which diagonalizes A with

$$(Q^T A Q)_{ii} = A_i\ .$$

Proof: For P given, let B be the matrix defined by

$$B = P^T D^d\ P$$

and whose eigenvalues are D_i. Then, from Lemma 4.2, we have:

$$\mathrm{Tr}\ [P^T D^d P\ A] = \mathrm{Tr}\ (AB) < \sum_{i=1}^{N} A_i\ D_i .$$

On the other hand, we also have

$$\mathrm{Tr}[QD^d Q^T A] = \mathrm{Tr}\ [Q^T QD^d Q^T AQ] = \mathrm{Tr}\ [D^d Q^T AQ) = \sum_{i=1}^{N} A_i D_i$$

and the result follows. □

We are now ready to prove the main result of this section, which reduces the local problems to convex minimization problems set on $R^N (N = 2 \text{ or } 3)$.

THEOREM 4.1: If the internal dissipation potential $\mathcal{D}_1(\cdot)$ *is convex and isotropic, then the solution* $\mathbf{H}$ *of the local problems* (4.1) *is given by*

$$\boxed{\begin{aligned} &\mathbf{H}\big|_{\Omega_\ell^i} = \mathbf{Q}_\ell \, \mathbf{H}_\ell^d \, \mathbf{Q}_\ell^T, \\ &\mathbf{A}_\ell^i = r\left\{ \frac{R}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^T) - \lambda \right\}\Big|_{\Omega_\ell^i} , \end{aligned}} \tag{4.3}$$

where $\mathbf{Q}_\ell$ *is an orthogonal matrix of* $\mathbf{R}^{N\times N}$ *whose columns are normed eigenvectors of the matrix* $\mathbf{A}_\ell^i$ (the first column corresponding to the biggest eigenvalue and so on) *and where* $\mathbf{H}_\ell^d$ *is the diagonal matrix of* $\mathbf{R}^{N\times N}$ *solution of*

$$\text{(4.4)} \quad \underline{\text{Minimize}} \ \{\mathcal{D}_1(\mathbf{D}^d) + \frac{Rr}{2} |\mathbf{D}^d|^2 - \sum_{i=1}^N A_i (\mathbf{D}^d)_{ii}\} \quad \text{on} \quad D^N \quad .$$

Above D^N *denotes the space of diagonal matrices of* $\mathbf{R}^{N\times N}$ *and* (A_i) *denotes the set of eigenvalues of* $\mathbf{A}_\ell^i$ $(A_1 \geq A_2 \geq \cdots \geq A_N)$.

Proof. First observe that any matrix $\mathbf{G}$ of $\mathbf{R}_s^{N\times N}$ can be decomposed into

$$\mathbf{G} = \mathbf{P}^T \mathbf{D}^d \mathbf{P} , \tag{4.5}$$

where $\mathbf{P}$ and $\mathbf{D}^d$ are two independent matrices of $\mathbf{R}^{N\times N}$, $\mathbf{P}$ being orthogonal $(\mathbf{P}\,\mathbf{P}^T = \mathbf{Id})$ and $\mathbf{D}^d$ being the diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{G}$. Then, if we denote by O^N (respectively D^N) the space of orthogonal (respectively diagonal) matrices of $\mathbf{R}^{N\times N}$, the local problem (4.1) becomes

$$\text{(4.6)} \quad \forall\, i,\ \forall \ell,\ \underline{\text{Minimize}} \ J_\ell^i(\mathbf{P},\mathbf{D}^d) \ \underline{\text{over}} \ O^N \times D^N,$$

with

$$J_\ell^i(\mathbf{P},\mathbf{D}^d) = J_\ell^i(\mathbf{G}) = \mathcal{D}_1^e(\mathbf{P}^T\mathbf{D}^d\mathbf{P}) + \frac{Rr}{2}|\mathbf{P}^T\mathbf{D}^d\mathbf{P}|^2 - (\mathbf{P}^T\mathbf{D}^d\mathbf{P}) \cdot \mathbf{A}_\ell^i.$$

But since $\mathcal{D}_1^e(\cdot)$ and $|\cdot|$ are isotropic $(\mathcal{D}_1^e(\mathbf{P}^T\mathbf{D}^d\mathbf{P}) = \mathcal{D}_1^e(\mathbf{D}^d))$, we can rewrite J_ℓ^i as

$$J_\ell^i(\mathbf{P},\mathbf{D}^d) = \mathcal{D}_1^e(\mathbf{D}^d) + \frac{Rr}{2} |\mathbf{D}^d|^2 - (\mathbf{P}^T\mathbf{D}^d\mathbf{P}) \cdot \mathbf{A}_\ell^i.$$

Now, let $\mathbf{H}_\ell^d$ be the solution of (4.4) (unique since the function to minimize is strictly convex) and let $\mathbf{P}_H$ be the permutation matrix which reorders the diagonal elements of $\mathbf{H}_\ell^d$ in the decreasing order. Since $A_1 > A_2 > \cdots > A_N$ and since $\mathcal{D}_1^e(\cdot)$ is isotropic, we have

$$\mathcal{D}_1^e(\mathbf{P}_H^T\mathbf{H}_\ell^d\mathbf{P}_H) + \frac{rR}{2}\,|\mathbf{P}_H^T\mathbf{H}_\ell^d\mathbf{P}_H|^2 - \sum_{i=1}^N A_i(\mathbf{P}_H^T\mathbf{H}_\ell^d\mathbf{P}_h)_i < \mathcal{D}_1^e(\mathbf{H}_\ell^d) + \frac{Rr}{2}\,|\mathbf{H}_\ell^d|^2 - \sum_{i=1}^N A_i(\mathbf{H}_\ell^d)_i \ .$$

But $\mathbf{H}_\ell^d$ is the only solution of (4.4), thus $\mathbf{P}_H^T\,\mathbf{H}_\ell^d\,\mathbf{P}_H$ has to be equal to $\mathbf{H}_\ell^d$, which means that the diagonal elements of $\mathbf{H}_\ell^d$ are already placed in a decreasing order. From Lemma 4.3, this implies

$$(4.7) \qquad \sum_{i=1}^N A_i(\mathbf{H}_\ell^d)_i = (\mathbf{Q}_\ell\,\mathbf{H}_\ell^d\,\mathbf{Q}_\ell^T)\cdot\mathbf{A}_\ell \quad .$$

Moreover, since $\mathbf{H}_\ell$ is solution of (4.4), we also have

$$\mathcal{D}_1^e(\mathbf{H}_\ell^d) + \frac{rR}{2}|\mathbf{H}_\ell^d| - \sum_{i=1}^N A_i(\mathbf{H}_\ell^d)_i < \mathcal{D}_1^e(\mathbf{D}^d) + \frac{Rr}{2}\,|\mathbf{D}^d|^2 - \sum_{i=1}^N A_i D_i,\ \forall\, \mathbf{D}^d \in D^N,$$

and in particular, if P_D is the permutation matrix reordering $\mathbf{D}^d$

$$\mathcal{D}_1^e(\mathbf{H}_\ell^d) + \frac{Rr}{2}\,|\mathbf{H}_\ell^d| - \sum_{i=1}^N A_i(\mathbf{H}_\ell^d)_i < \mathcal{D}_1^e(\mathbf{P}_D^T\mathbf{D}^d\mathbf{P}_d) + \frac{Rr}{2}\,|\mathbf{P}_D^T\mathbf{D}^d\mathbf{P}_D|^2 - (\mathbf{P}^T\mathbf{D}^d\mathbf{P}_D)_i A_i,\ \forall\, \mathbf{D}^d.$$

Now, from (4.7), Lemma 4.3, and the isotropy of $\mathcal{D}_1^e(\cdot)$, this implies

$$\mathcal{D}_1^e(\mathbf{H}_\ell^d) + \frac{Rr}{2}\,|\mathbf{H}_\ell^d|^2 - (\mathbf{Q}_\ell\mathbf{H}_\ell^d\mathbf{Q}_\ell^T)\cdot(\mathbf{A}_\ell^i) < \mathcal{D}_1^e(\mathbf{D}^d) + \frac{Rr}{2}\,|\mathbf{D}^d|^2 - (\mathbf{P}^T\mathbf{D}^d\mathbf{P})\cdot(\mathbf{A}_\ell^i) \quad ,$$

for any $\mathbf{D}^d$ in D^N and any $\mathbf{P}$ in O^N. So, finally, we get

$$J_\ell^i\,(\mathbf{Q}_\ell,\mathbf{H}_\ell^d) < J_\ell^i(\mathbf{P},\mathbf{D}^d) \quad \forall(\mathbf{P},\mathbf{D}^d)\ ,\ \text{in}\ O^N \times D^N \ .$$

In other words, $\{\Omega_\ell, H_\ell^d\}$ is a solution of (4.6). By construction, this implies that H, given by (4.3), is a solution of the original local problems (4.1). Our proof is therefore complete because, since J_ℓ^i is strictly convex, such a solution is unique. $\square$

<u>REMARK 4.1</u>. It is well known that the internal dissipation potential is isotropic and convex for all standard isotropic viscoplastic solids and all standard viscoplastic fluids. Theorem 4.1 can therefore be applied in most practical situations. $\square$

<u>REMARK 4.2</u>: In Theorem 4.1, the relation (4.3) simply expresses that $H|_{\Omega_\ell^i}$ and A_ℓ^i have the same eigenvectors. In essence, this is the discrete equivalent of the well known result which states that principal stresses and principal strains are parallel in isotropic elasticity or visoplasticity. $\square$

5. NORTON VISCOPLASTICITY

<u>5.1 The local problems</u>. Norton viscoplasticity corresponds to one of the easiest possible case where the internal dissipation potential $\mathcal{D}_1(\cdot)$ is given by

$$(5.1) \qquad \mathcal{D}_1(G) = \frac{1}{p}(k\sqrt{2})^p |G|^p = \frac{1}{p}(k\sqrt{2})^p \left(\sum_{i,j=1}^{N} G_{ij}^2\right)^{p/2} .$$

Each local minimization problem (4.1) now becomes

(5.2) <u>Minimize</u> $J_\ell^i(G)$ <u>over</u> $\mathbf{R}_s^{N\times N}$ <u>with</u>

$$\begin{cases} J_\ell^i(G) = \frac{1}{p}(k\sqrt{2})^p |G|^p + \frac{Rr}{2}|G|^2 - A_\ell^i \cdot G, \\ A_\ell^i = r\{\frac{R}{2}(\nabla v + \nabla v^T) - \lambda\}|_{\Omega_\ell^i} , \end{cases}$$

whose solution is given by

THEOREM 5.1: The solution B_ℓ^i of the local minimization problem (5.2) is of the form

(5.3) $B_\ell^i = A_\ell^i x/|A_\ell^i|$,

where x is the solution of the one-dimensional convex minimization problem

(5.4) Minimize $\{\frac{1}{p}(k\sqrt{2})^p y^p + \frac{Rr}{2} y^2 - |A_\ell^i|y\}$ over R_+.

Proof: Suppose that the norm $|B_\ell^i|$ of the minimizer B_ℓ^i of J_ℓ^i over R_s^{NxN} is known. Then, B_ℓ^i minimizing J_ℓ^i over R_s^{NxN} will in particular minimize $J_\ell^i(\cdot)$ over the set of matrices of R_s^{NxN} with fixed norm $|B_\ell^i|$. But this last minimization problem reduces to the maximization of the scalar product $A_\ell^i \cdot G$ over a sphere of R_s^{NxN} and its solution is given by

(5.3) $$B_\ell^i = A_\ell^i\ |B_\ell^i|/|A_\ell^i|.$$

By plugging (5.3) into the expression of J_ℓ^i, the minimization of J_ℓ^i over R_s^{NxN} finally reduces to finding the unknown norm $|B_\ell^i|$ which has to mininimize $J_\ell^i(A_\ell^i y/|A_\ell^i|)$ over the set R_+ of positive numbers. This last problem is precisely (5.4) and our proof is complete. □

In practice, the numerical solution of each local problem (5.2) uses Theorem 5.1 and is achieved by:

a) the computation of the solution $|B_\ell^i|$ of the one dimensional convex minimization problem (5.4)

b) the computation of B_ℓ^i by the explicit formula (5.3).

The numerical solution of (5.4) can be achieved for example by using the one dimensional Newton algorithm below

data: x^o = solution of (4.4) at previous iteration;	
initialisation:	$x \leftarrow x^o$;
	$j \leftarrow 1$;
repeat :	$j \leftarrow j + 1$;
	$g \leftarrow (k\sqrt{2})^p\, x^{p-1} + rRx - \lvert \mathbf{a}_\ell + \mathbf{a}_\ell^T \rvert / 2$;
	test : if $\lvert g \rvert$ below tolerance exit;
	$dg \leftarrow (p-1)(k\sqrt{2})^p\, x^{p-2} + Rr$;
	$x \leftarrow \max\,(10^{-30},\, x - (dg)^{-1} g)$;
exit	$\lvert \mathbf{H}_\ell \rvert = x$.

Putting together all the steps which permit the numerical solution of the decomposed approximated viscous flow problem (3.6) by the algorithm (3.11)-(3.14), we finally obtain the simple and easy to code computer flow chart of Figure 5.1.

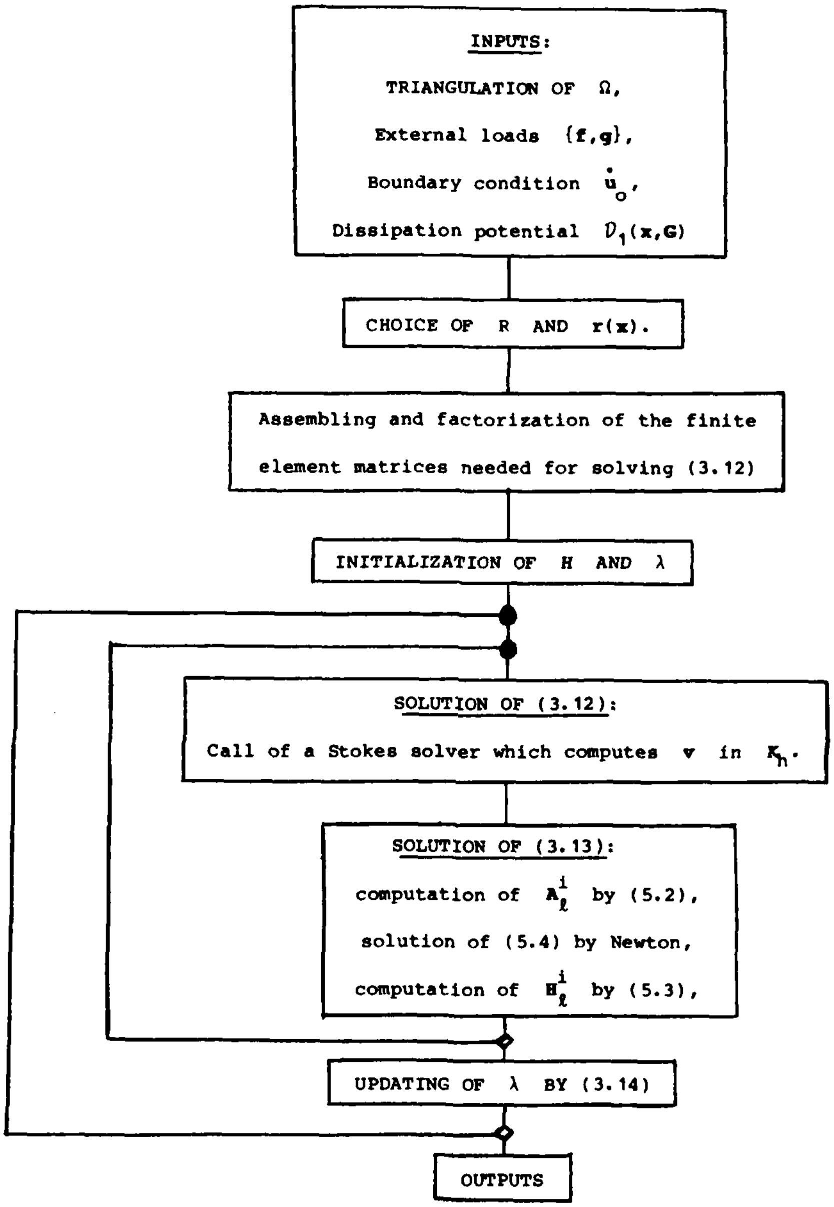


Fig. 5.1: Computer flow chart for solving Norton viscous flow problems

5.2 Numerical results. The first numerical test considers a horizontal cylindrical hose, with external radius 1., which is glued on a rigid core on its internal face, and which is subjected to its own weight. This situation may represent for example the cooling process of the plastic coating of an electrical wire, for which manufacturers must verify that the deformations undergone by the coating during cooling remain small. In a first approximation, strains are assumed to remain plane, and the coating is supposed to be made of an homogeneous Norton material with $k = .47$, $p = 1.4$, and volumic weight .1.

For symmetry reasons, only the right half of the section is considered; 225 nodes are used to approximate the velocity, 65 nodes are used for the pressure. Only one block-relaxation iteration is done at each Uzawa step and after 60 iterations of the Uzawa algorithm, the error $\|H^n - D(v^n)\|$ has decreased by a factor of 10^{-7} (in fact, 30 iterations were more than sufficient to obtain a very accurate velocity field). The total CPU time was approximatively 3mn on the VAX 780. Figure 5.2 represents the computed velocity field (magnified 7 times). The shape of the hose after one second of flow (the deformations being multiplied by 40) is indicated on Figure 5.3, together with the mesh used for the pressure.

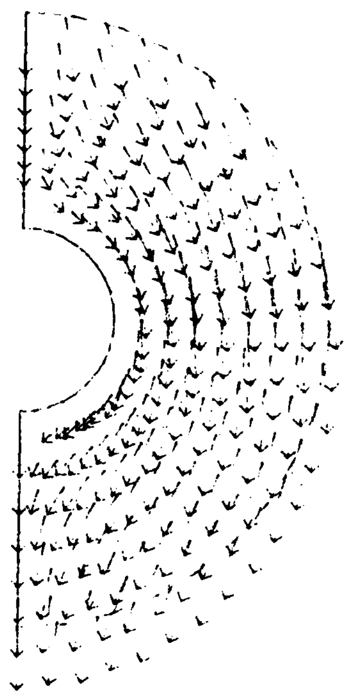
In the entire computation, the parameter R was equal to 1. The weight $r(x)$ was equal to 1 during the first 20 iterations, then updated by the formula $r(x) = |H^{20}|^{p-2}$ (see Remark 3.2 for justification), kept that way until iteration 40 where it was finally updated by $r(x) = |H^{40}|^{p-2}$.

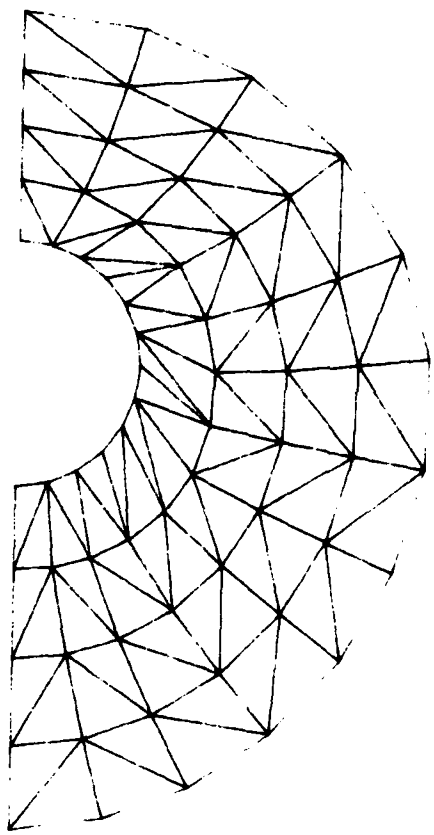
6. BINGHAM VISCOPLASTICITY.

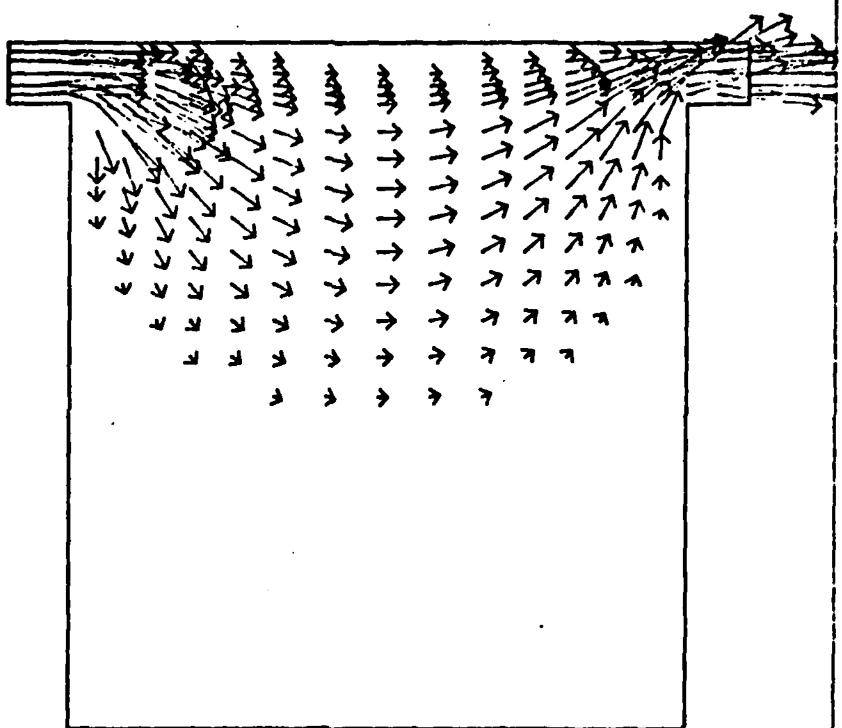
6.1 The local problems. Bingham viscoplasticity corresponds to an internal dissipation potential $\mathcal{D}_1(\cdot)$ of the type

(6.1) $$\mathcal{D}_1(G) = \mu|G|^2 + g\sqrt{2}\ |G|.$$

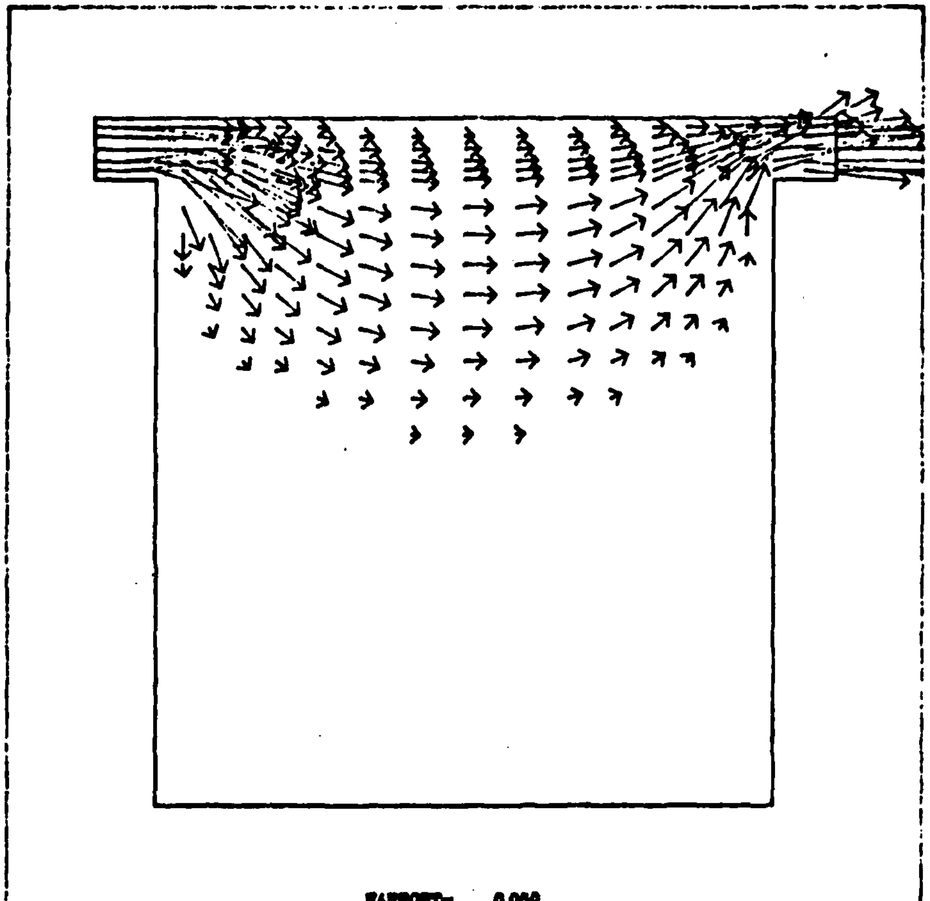
Each local minimization problem (4.1) now becomes







RAPPORT= 0.060
BINGHAM FLUID IN A CAVITY



RAPPORT= 0.060
BINGHAM FLUID IN A CAVITY

$$(6.2)\quad \underline{\text{Minimize}}\quad \{(\frac{rR}{2}+\mu)|G|^2 + g\sqrt{2}\ |G| - A_\ell^i \cdot G\}\quad \underline{\text{over}}\quad R_s^{N\times N}$$

whose solution B_ℓ^i is simply given by the explicit formula

$$(6.3)\qquad B_\ell^i = \text{Max}\ \{0, 1-\sqrt{2}\ g/|A_\ell^i|\}\ \ A_\ell^i\ /(rR+2\mu)\ .$$

In practice, the whole program solving the flow problem (3.6) for Bingham fluids still corresponds to the computer flow chart of Fig. 5.1, each local problem being now solved by (6.3).

6.2 Numerical result. We consider herein a Bingham fluid flowing viscously through a cavity. Fluid enters at the upper right of the cavity and exits at the upper left, with an imposed velocity of 3.0. No slip boundary conditions are imposed elsewhere. Dimensions of the cavity are 1. for the main square and .1 for the entrance and exit tubes. The fluid viscosity μ is assumed to be .01 and velocities are supposed to remain plane.

The finite element mesh uses 419 nodes for velocities and 166 nodes for pressures. One block-relaxation is done at each Uzawa step and we take $R = .1$ and $r(x) = 1.0$. Figures 6.1 and 6.2 represent velocity obtained after 40 iterations, the plasticity threshold $g\sqrt{2}$ being respectively of μ and of 10μ. As expected, the domain where the fluid is at rest is bigger in the latter case. The computation time for each case was approximatively 10 mn on the VAX 780.

7. TRESCA TYPE VISCOPLASTICITY IN PLANE STRESSES.

7.1 The local problems. In plane stresses, the body under consideration is supposed to be very thin along x_3 and is loaded in its plane so that, in a first approximation, all stresses along x_3 are equal to zero. It is then possible to eliminate the x_3 direction and to reduce our original problem to a two-dimensional one whose domain will be the middle plane section of the body and whose unknowns will be the in-plane velocities.

These inplane velocities need no longer be incompressible since any reduction of the plane section can be compensated by a corresponding thickening of the body. Therefore, in Sections 2 and 3, K_h is everywhere replaced by $V_h + \dot{u}_o$ and in particular (3.12) becomes

$$\underline{\text{Minimize}} \qquad L_R(\cdot,G,\lambda) \ \underline{\text{over}} \ V_h + \dot{u}_o ,$$

which is a classical linear elasticity problem with a zero first Lamé coefficient. As for (3.13), its formulation is unchanged and it still reduces to the local problem (4.1).

In plane stresses, a Tresca type viscoplastic material corresponds to the internal dissipation potential

$$\mathcal{D}_1(G) = \frac{1}{p}\,(k\sqrt{2})^p\ \{\mathrm{Max}(|G_1|,|G_2|,|G_1+G_2|)\}^p \tag{7.1}$$

where G_1 and G_2 are the eigenvalues of the 2×2 symmetric matrix G. This potential is a symmetric convex function of the eigenvalues of G, therefore is isotropic and convex (HILL[1970]). Moreover it satisfies the inequalities (1.6) for coerciveness and continuity. On the other hand, this potential is not strictly convex and not differentiable, which clearly appears in Fig. 7.1, where the level lines of $\mathcal{D}_1$ are drawn.

The viscous flow problem associated to this potential (7.1) can still be solved by the Uzawa algorithm (3.11)-(3.14); (3.12) is a linear elasticity problem and (3.13) reduces to local problems whose solution H is given by (see Theorem 4.1).

$$(7.2)\quad \left\{\begin{aligned}
& H\big|_{\Omega^i_\ell} = Q_\ell H^d_\ell\, Q^T_\ell , \\
& A^i_\ell = r\left(\frac{R}{2}\,(\nabla v+\nabla v^T) - \lambda\right)\Big|_{\Omega^i_\ell} , \quad H^d_\ell = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} , \\
& j\,(H_1,H_2) \le j\,(D_1,D_2), \quad \forall\,\{D_1,D_2\} \in R^2 , \\
& j\,(D_1,D_2) = \frac{1}{p}\,(k\sqrt{2})^p\left[\mathrm{Max}\,\{|D_1|,|D_2|,|D_1+D_2|\}\right]^p + \frac{rR}{2}\,(D_1^2 + D_2^2) - A_1 D_1 - A_2 D_2 .
\end{aligned}\right.$$

Above, as before, Λ_1 and Λ_2 ($\Lambda_1 \geq \Lambda_2$) denote the eigenvalues of $\mathbf{A}_\ell^i$ and $\mathbf{Q}_\ell$ is an orthogonal transformation matrix which diagonalizes $\mathbf{A}_\ell^i$ and orders the diagonal elements of the resulting matrix in a decreasing order.

Thus, for Tresca type viscoplasticity in plane stresses, the numerical solution of each local problem (4.1) reduces to

<u>Step 1</u> <u>compute</u> $\mathbf{A}_\ell^i = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix}$ <u>by</u> $\mathbf{A}_\ell^i = r(\frac{R}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T) - \lambda)|_{|\Omega_\ell^i}$;

<u>Step 2</u> <u>compute the eigenvalues</u> Λ_1 <u>and</u> Λ_2 <u>by</u>

$$\Lambda_1 = \left[(A_{11}+A_{22}) + \sqrt{(A_{11}-A_{22})^2 + 4A_{12}^2} \right]/2 ,$$

$$\Lambda_2 = \left[(A_{11}+A_{22}) - \sqrt{(A_{11}-A_{22})^2 + 4A_{12}^2} \right]/2 ;$$

<u>Step 3</u> <u>compute</u> H_1 <u>and</u> H_2 <u>by</u>

$j(H_1,H_2) \leq j(D_1,D_2), \ \forall \{D_1,D_2\} \in R^2;$

<u>Step 4:</u> <u>Compute</u> $H_\ell^i = \mathbf{Q}_\ell \, \mathbf{H}_\ell^d \, \mathbf{Q}_\ell^T$ <u>by</u>

$$(\mathbf{H}_\ell^i)_{11} = (H_1-H_2)\,(A_{11}-\Lambda_2)/(\Lambda_1-\Lambda_2) + H_2,$$
$$(\mathbf{H}_\ell^i)_{22} = (H_1-H_2)(A_{22}-\Lambda_1)/(\Lambda_1-\Lambda_2) + H_1,$$
$$(\mathbf{H}_\ell^i)_{12} = (H_1-H_2)\,A_{12} \;/\; (\Lambda_1-\Lambda_2).$$

Consequently, the computer flow chart associated to the Uzawa algorithm (3.11)-(3.14) for the numerical solution of viscous flow problems in Tresca type viscoplasticity in plane stresses is the one described in Figure 5.1, but with the solution of the local problems being achieved by the four steps above. Among these steps, only one, Step 3, is not explicit and its solution is described in the next paragraph.

7.2 Solution of Step 3. By definition of the subgradient, Step 3 is equivalent to

$$\{0,0\} \in \partial j(H_1,H_2). \tag{7.3}$$

Therefore, to solve Step 3, we first begin by computing the subgradient of $j(\cdot,\cdot)$ over $\mathbb{R}^2$. Since we know from Theorem 4.1 that, at the solution of (7.3), H_1 is greater or equal to H_2, we restrict ourselves to the half plane $D_1 > D_2$. Then, a direct calculation gives:

Case 1: if $\{D_1,D_2\} \in K_1 = \{\{x,y\} \in \mathbb{R}^2,\ x > y,\ y > 0\}$, then:

$$\begin{cases} j(D_1,D_2) = \frac{1}{p}(k\sqrt{2})^p(D_1+D_2)^p + \frac{rR}{2}(D_1^2+D_2^2) - A_1D_1 - A_2D_2, \\ \partial j(D_1,D_2) = \{(u_1,u_2),\ u_i = (k\sqrt{2})^p(D_1+D_2)^{p-1} + rR\,D_i - A_i\}. \end{cases}$$

Case 2: if $\{D_1,D_2\} \in K_2 = \{\{x,y\} \in \mathbb{R}^2,\ x > 0,\ y = 0\}$, then:

$$\begin{cases} j(D_1,D_2) = \frac{1}{p}(k\sqrt{2})^p(D_1)^p + \frac{rR}{2}D_1^2 - A_1D_1, \\ \partial j(D_1,D_2) = \{(u_1,u_2),\ u_1 = k\sqrt{2}(D_1)^{p-1} + rR\,D_1 - A_1,\ -A_2 < u_2 < (k\sqrt{2})^p(D_1)^{p-1} - A_2\}. \end{cases}$$

Case 3: if $\{D_1,D_2\} \in K_3 = \{\{x,y\} \in \mathbb{R}^2,\ x > -y,\ y < 0\}$, then

$$\begin{cases} j(D_1,D_2) = \frac{1}{p}(k\sqrt{2}\,D_1)^p + \frac{rR}{2}(D_1^2+D_2^2) - A_1D_1 - A_2D_2, \\ \partial j(D_1,D_2) = \{(u_1,u_2),\ u_i = (k\sqrt{2})^p(D_1)^{p-1}\,\delta_{i1} + rR\,D_i - A_i\}. \end{cases}$$

Case 4: if $\{D_1,D_2\} \in K_4 = \{\{x,y\} \in \mathbb{R}^2,\ x = -y,\ y < 0\}$, then

$$\begin{cases} j(D_1,D_2) = \frac{1}{p}(-k\sqrt{2}D_2)^p + \frac{rR}{2}(D_1^2+D_2^2) - A_1D_1 - A_2D_2, \\ \partial j(D_1,D_2) = \{(u_1,u_2),\ rRD_1 - A_1 < u_1 < (k\sqrt{2})^p(D_1)^{p-1} + rRD_1 - A_1, \\ \qquad rRD_2 - A_2 - (k\sqrt{2})^p(-D_2)^{p-1} < u_2 < rRD_2 - A_2\}. \end{cases}$$

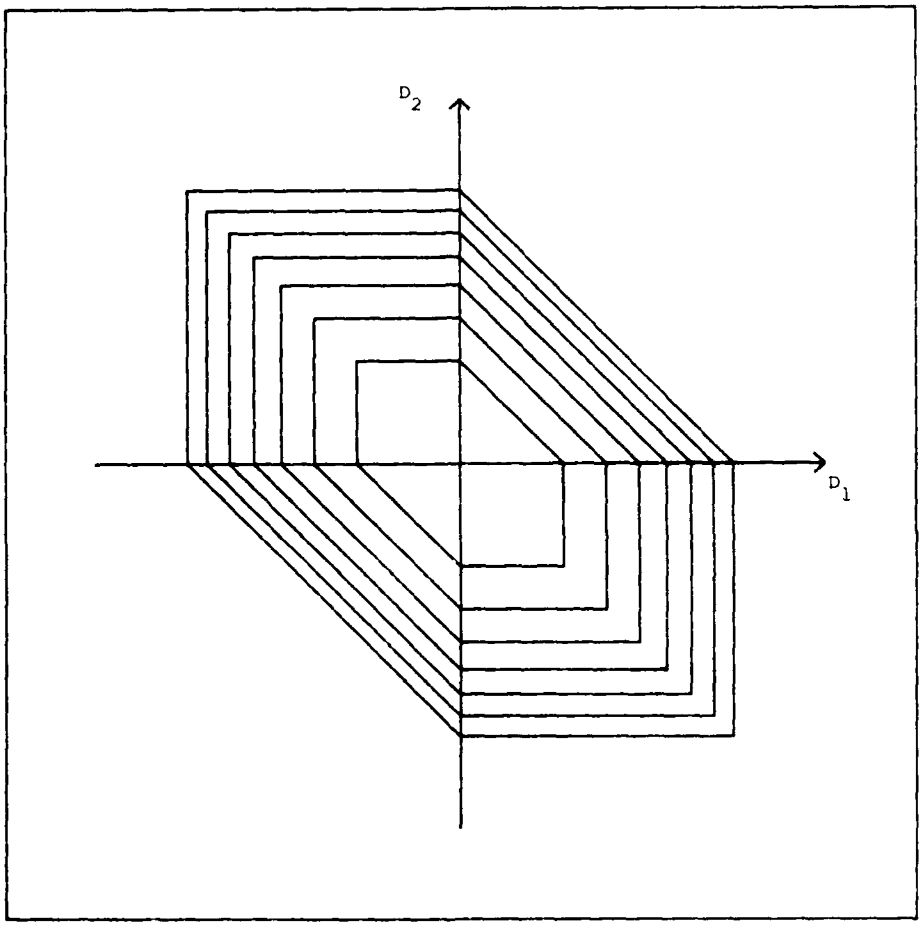


Figure 7.1: level lines of $\mathcal{D}_1$. (p=2)

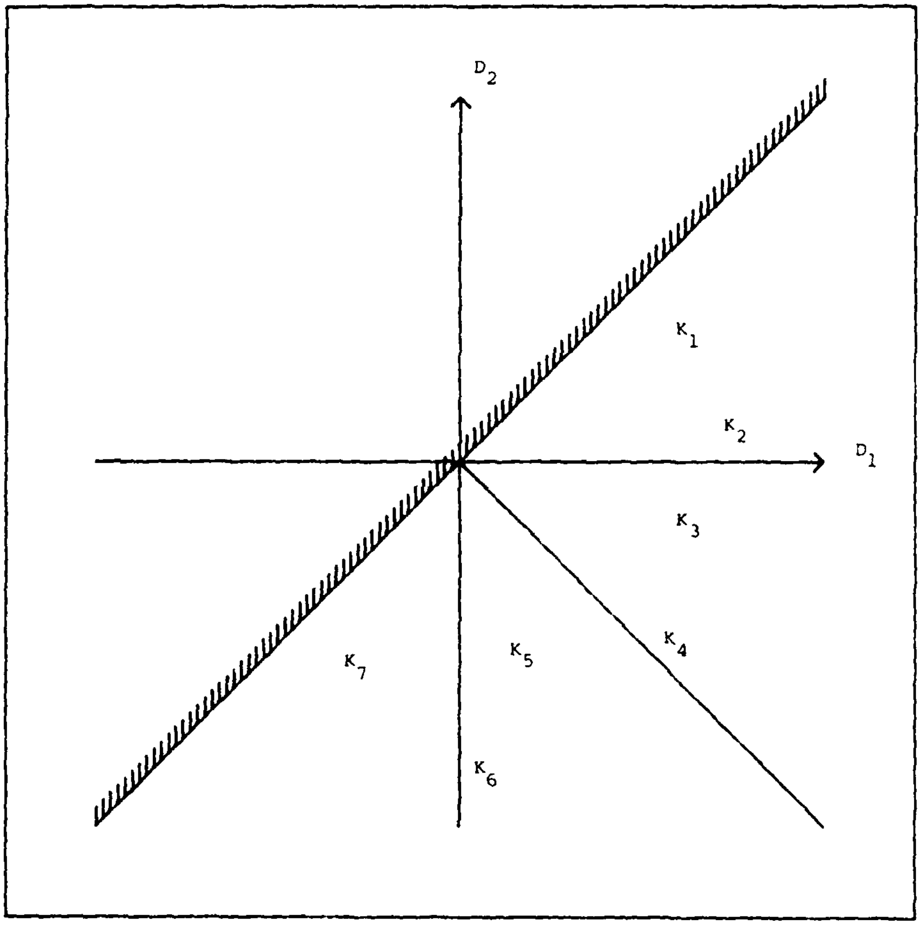


Figure 7.2: Partition of the half plane $D_1 \geq D_2$.

Case 5: if $\{D_1,D_2\} \in K_5 = \{\{x,y\} \in R^2,\ -y > x > 0\}$, then

$$\begin{cases} j(D_1,D_2) = \frac{1}{p}(k\sqrt{2})^p(-D_2)^p + \frac{Rr}{2}(D_1^2+D_2^2) - A_1D_1 - A_2D_2, \\ \partial j(D_1,D_2) = \{(u_1,u_2),\ u_i = -(k\sqrt{2})^p(-D_2)^{p-1}\,\delta_{i2} + rR\,D_i - A_i\}. \end{cases}$$

Case 6: if $\{D_1,D_2\} \in K_6 = \{\{x,y\} \in R^2,\ x = 0,\ y < 0\}$, then

$$\begin{cases} j(D_1,D_2) = \frac{1}{p}(k\sqrt{2})^p(-D_2)^p + \frac{rR}{2}D_2^2 - A_2D_2, \\ \partial j(D_1,D_2) = \{(u_1,u_2), -(k\sqrt{2})^p(-D_2)^{p-1} - A_1 < u_1 < -A_1, u_2 = -(k\sqrt{2})^p(-D_2)^{p-1} + rRD_2 - A_2\}. \end{cases}$$

Case 7: if $\{D_1,D_2\} \in K_7 = \{\{x,y\} \in R^2,\ 0 > x > y\}$, then

$$\begin{cases} j(D_1,D_2) = \frac{1}{p}(k\sqrt{2})^p(-D_1-D_2)^p + \frac{Rr}{2}(D_1^2+D_2^2) - A_1D_1 - A_2D_2, \\ \partial j(D_1,D_2) = \{(u_1,u_2),\ u_i = -(k\sqrt{2})^p(-D_1-D_2)^{p-1} + rR\,D_i - A_i\}. \end{cases}$$

Observe that the sets K_i form a partition of the half plane $D_1 > D_2$. Then, by definition of (7.3) and since its solution $\{H_1,H_2\}$ belongs to this halfplane (Theorem 4.1), we have:

(7.4) $$\{H_1,H_2\} = \bigcup_{i=1}^{7} \{\{x_i,y_i\} \in K_i,\ \{0,0\} \in \partial j(x_i,y_i)\}.$$

Therefore the solution of (7.3) is simply the solution of one of the local subproblems (the one which admits a solution for the given data of A_1 and A_2)

$$\{0,0\} \in \partial j(x_i,y_i),\quad \{x_i,y_i\} \in K_i.$$

Then, once that for each subproblem the conditions which guarantee the existence of solutions are explicited and that the algebraic expressions of these solutions are computed, $\{H_1,H_2\}$ is simply obtained by:

(i) finding which subproblem has a solution for the given values of A_1 and A_2, by checking successively the admissibility requirements (conditions for existence of solutions) of each subproblem;

(ii) setting $\{H_1, H_2\}$ equal to the corresponding solution.

Here, these computations are easy to carry out since we have just computed the algebraic expressions of $\partial j(\cdot,\cdot)$ on each subset K_i. For example, for $i = 1$, we have

local subproblem:

$$\begin{cases} (k\sqrt{2})^p (x+y)^{p-1} + rRx - A_1 = 0, \\ (k\sqrt{2})^p (x+y)^{p-1} + rRy - A_2 = 0, \\ x > y > 0; \end{cases}$$

admissibility requirement (necessary and sufficient condition for existence of solutions):

$$A_2 > [(A_1 - A_2)/rR]^{p-1}$$

solution

$$\begin{cases} x_1 = (z+(A_1-A_2)/rR)/2, \ y_1 = (z-(A_1-A_2)/rR)/2, \\ \\ z \ \underline{\text{minimizes}} \ \{\frac{2}{p} (k\sqrt{2})^p t^p + \frac{rR}{2} t^2 - (A_1+A_2)t\} \ \underline{\text{over}} \ R_+. \end{cases}$$

All computations done, the solution $\{H_1, H_2\}$ of Step 3 is finally given by:

$$
\text{for } A_2 > ((A_1-A_2)/rR)^{p-1} \quad
\begin{cases}
H_1 = [z+(A_1-A_2)/rR]/2, \\
H_2 = [z-(A_1-A_2)/rR]/2, \\
z \ \underline{\text{minimizes}} \ \{\frac{2}{p}(k\sqrt{2})^p t^p + \frac{Rr}{2}t^2 - (A_1+A_2)t\} \ \ \underline{\text{over}} \ \ R_+,
\end{cases}
$$

$$
\text{for } ((A_1-A_2)/rR)^{p-1} > A_2 > 0 \quad
\begin{cases}
H_1 \ \underline{\text{minimizes}} \ \{\frac{1}{p}(k\sqrt{2})^p t^p + \frac{Rr}{2}t^2 - A_1 t\} \ \ \text{over} \ \ R_+, \\
H_2 = 0 \ ;
\end{cases}
$$

$$
\text{for } A_1 > (-A_2 + (-A_2/Rr)^{p-1}) > 0
\begin{cases}
H_1 \ \underline{\text{minimizes}} \ \{\frac{1}{p}(k\sqrt{2})^p t^p + \frac{Rr}{2}t^2 - A_1 t\} \ \ \text{over} \ \ R_+, \\
H_2 = A_2/Rr;
\end{cases}
$$

$$
\begin{array}{l}
\text{for } (-A_2+(-A_2/rR)^{p-1}) > A_1 > 0 \\
\text{and } (A_1/rR)^{p-1} + A_1 > -A_2
\end{array}
\begin{cases}
H_1 \ \underline{\text{minimizes}} \{\frac{1}{p}(k\sqrt{2})^p t^p + Rrt^2 - (A_1-A_2)t\} \ \text{over} \ R_+, \\
H_2 = -H_1;
\end{cases}
$$

$$
\text{for } -A_2 > A_1 + (A_1/rR)^{p-1} > 0
\begin{cases}
H_1 = A_1/rR \\
(-H_2) \ \underline{\text{minimizes}} \{\frac{1}{p}(k\sqrt{2})^p t^p + Rrt^2 + A_2 t\} \ \ \underline{\text{over}} \ \ R_+;
\end{cases}
$$

$$
\text{for } 0 > A_1 > -((A_1-A_2)/rR)^{p-1}
\begin{cases}
H_1 = 0 \\
(-H_2) \ \underline{\text{minimizes}} \ \{\frac{1}{p}(k\sqrt{2})^p t^p + \frac{Rr}{2}t^2 + A_2 t\} \ \ \text{over} \ \ R_+;
\end{cases}
$$

$$
\text{for } -((A_1-A_2)/rR)^{p-1} > A_1
\begin{cases}
H_1 = (z+(A_1-A_2)/rR)/2, \\
H_2 = (z-(A_1-A_2)/rR)/2, \\
(-z) \ \underline{\text{minimizes}} \ \{\frac{2}{p}(k\sqrt{2})^p t^p + \frac{rR}{2}t^2 + (A_1+A_2)t\} \ \ \text{over} \ \ R_+.
\end{cases}
$$

In the above formulas, the minimization over R_+ is numerically achieved by using the one dimensional Newton algorithm described in Section 5 of this report.

7.3 Numerical result. We now consider a perforated square thin plate (width = 1.), subjected to an uniform traction of .52 per unit area on two of its opposite faces. This plate is supposed to be made of a Tresca material with $p = 1.5$ and $k = 1/\sqrt{2}$.

For symmetry reasons, only one fourth of the plate is considered. On this fourth, 126 nodes are used for approximating the velocity field. One block-relaxation iteration is done per Uzawa step, and the parameter R and the weight $r(x)$ are respectively given by $R = 1$ and $r(x) = |H|^{p-2}$, H being the deformation rate tensor corresponding to a computation done on the same geometry but with $\mathcal{D}_1(G) = \frac{1}{p}(k\sqrt{2})^p|G|^p$ (compressible Norton material in plane strains).

After 50 iterations, the error $\|D(v^n) - H^n\|$ is decreased by a factor of 10^{-4}, and the total dissipated energy rate is equal to -2.737 for the whole plate. The corresponding velocitites are indicated on Fig 7.2. It must be noticed that, due to the little number of boundary conditions imposed on v, this case is particularly unstable for most numerical methods.

8 POSSIBLE EXTENSIONS OF THE METHOD.

Many extensions can be considered for the numerical method described in this report. For example,

(i) different finite elements can be considered in the approximation K_h of the set of kinematically admissible incompressible velocity fields. Any finite element which is used with some success in the approximation of the Stokes problem can be employed here. Nevertheless, if the gradients of the elements of K_h are not piecewise constant, a numerical integration rule will be necessary to compute the dissipation $F(G)$, which leads to an additional truncation error and which slightly complicates Problem (3.6). Moreover, the space Y_h, which is then the space of functions which are characterized by

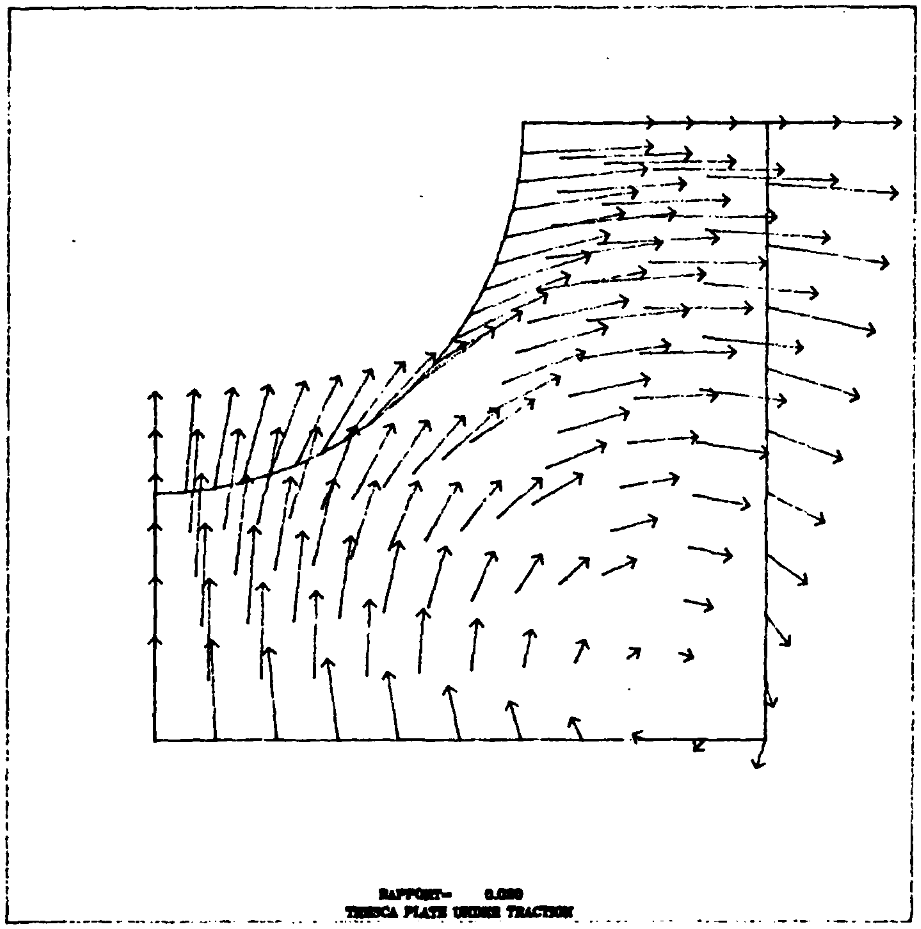


Figure 7.3

their values at the integration points, must be sufficiently large to contain, within an isomorphism, the image of K_h by the operator $D(\cdot)$;

(ii) an inertia term can be added in the formulation of the virtual work theorem. Through an implicit time discretization, the resulting problem will then reduce to a sequence of augmented lagrangian problems (3.6) (one per time step), the functional G being now replaced by

$$G(\mathbf{w}) = -\int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx - \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{w}\,da + \frac{1}{2DT}\int_\Omega \rho\,|\mathbf{w} - \mathbf{v}_n|^2\,dx.$$

Each problem (3.6) can still be solved by Algorithm (3.11) - (3.14). Problem (3.12) will again correspond to a linear Stokes type problem, associated to fixed, symmetric, positive definite finite element matrices. Problem (3.13) remains unchanged;

(iii) a convection term $\rho(\mathbf{v}\cdot\nabla)\mathbf{v}$ can also be added in the formulation of the virtual work theorem. Since the operator in $\mathbf{v}$ will no longer be self-adjoint, no augmented lagrangian L_R can then be introduced. Nevertheless, Algorithm (3.11) - (3.14) is still applicable there (FORTIN-GLOWINSKI [1982, p 71]. Problem (3.13) is unchanged, and (3.12) becomes

$$\int_\Omega r(x)(R(\mathbf{D}(\mathbf{v})-\mathbf{H})-\lambda)\cdot\mathbf{D}(\mathbf{w})\,dx + \int_\Omega \rho(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx + \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{w}\,da,\ \forall\,\mathbf{w}\in K_h.$$

For small convection terms, $\mathbf{v}$ can be replaced in the convection term by the solution $\mathbf{v}_k^n$ at the previous iterate, and (3.12) then reduces to an ordinary Stokes problem. For large convection terms, one can use optimal control techniques ([GLOWINSKI-LE TALLEC [1983]). All this is described in details by TANGUY [1983] which uses augmented lagrangian techniques in a very similar situation;

(iv) finally, our problem can be coupled to an heat diffusion problem if we suppose, for example, that the internal dissipation potential $\mathcal{D}_1(x, G)$ is a function of the temperature T at x. If convection phenomenon are not dominant, temperatures and velocities can be efficiently computed by block-relaxation: assuming the velocity to be given, one computes the temperature by solving the energy equation; then, assuming the

temperature to be given, the velocity is determined by solving (3.6), the process being repeated until convergence. Observe that, despite a possible change of the temperature field between two successive resolutions of (3.6), the finite element matrices do not have to be changed because the temperature is only a parameter in the local problems (4.1). This results in considerable economy in computer running time.

References

BERCOVIER, M., PIRONNEAU, O. [1977]: Estimation d'erreurs pour la resolution d'un probleme de Stokes en elements finis conformes de Lagrange. CRAS Paris, T286A, p 1085-1087.

BREZZI, F. [1974]: On the existence, uniqueness and approximation of saddlepoint problems arising from Lagrange multipliers. RAIRO Anal. Numer. 8R2, p 129-151.

CIARLET, P. G. [1978]: The finite element method for elliptic problems. Amsterdan, North Holland.

EKELAND, I., TEMAN, R. [1978]: Convex analysis and variational problems. Amsterdan, North Holland.

FORTIN, M., GLOWINSKI, R. [1982] (eds): Methodes de lagrangien augmente. Dunod-Bordas, Paris.

GEYMONAT, G., SUQUET, P. [1983]: Espaces fonctionnels pour les milieux de. Norton-Hoff (to appear).

GIRAULT, V., RAVIART, P. A. [1979]: Finite element approximation of the Navier-Stokes equations. Berlin, Heidelberg, New York. Springer Verlag.

GLOWINSKI, R. [1984]: Numerical methods for nonlinear variational problems. Berlin, Heidelberg, New York. Springer Verlag.

GLOWINSKI, R., LE TALLEC, P. [1983]: Numerical solution of partial differential equation problems in nonlinear mechanics by quadratic minimization methods. Conference on large scale scientific computing. Madison May 16-18, 1983. Academic Press.

GLOWINSKI, R., LIONS, J. L., TREMOLIERES, R. [1981]: Numerical Analysis of Variational

HILL, R. [1970]: Constitutive inequalities for isotropic elastic solids under finite strain. Proc. Roy. Soc. London A314 p 457-472.

LIONS, P. L., MERCIFP, B. [1979]: Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal., 16, p 964-979.

PFRZYNA [1966]: Fundamental problems in plasticity. Adv. Appl. Mech. 9, p 243-377.

SCHEURER, B. [1977]: Existence et approximation de points-selles pour certains problemes non lineaires. RAIRO, serie rouge, Anal. Numer., 11, p 369-400.

TANGUY, P. [1983]: Numerical Simulation of a Pseudo 3-D Turbulent Flow in a Kaplan Turbine Diffuser. Technical Report TR-83-10. Department of Chemical Engineering, University of Nova Scotia.

TAYLOR, C., HOOD, P. [1973]: A numerical solution of the Navier-Stokes equations using the finite element technique. Computers and Fluids, 1, p 73-100.

VON NEUMANN, J. [1937]: Some matrix inequalities and metrization of matric-space, Tomsk Univ. Rev. 1, p 286-300. Reprinted in collected works, Vol iv, Pergamon, Oxford, 1962.

PL/jgb

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2690	2. GOVT ACCESSION NO. AD-A142793	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Numerical Solution of Viscoplastic Flow Problems by Augmented Lagrangians		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Patrick Le Tallec		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Scientific Computing
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE May 1984
		13. NUMBER OF PAGES 59
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Viscoplasticity, convexity, incompressibility, finite elements, augmented lagrangians

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report describes an application of Augmented Lagrangian techniques to the numerical solution of quasistatic flow problems in incompressible viscoplasticity, focusing on cases where the internal viscoplastic dissipation potential is not a differentiable function of the material deformation rate. The stresses of elastic origin are neglected, and the variational formulation of these problems is approximated via mixed finite elements of order 1. Convergence results are proved or recalled, both for the finite element approximation and for the augmented lagrangian algorithm. A detailed study of

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DATE

ILME